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Optimal dynamic marketing-mix policies for frequently purchased products and services versus consumer durable goods: A generalized analytic approach

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ABSTRACT

This paper deals with the qualitative characterization of optimal pricing and advertising policies together with the optimal ratio of the advertising elasticity of demand to its price elasticity over time. The problem is studied for frequently purchased products and services (FPS) as well as consumer durable goods (CDG) in both monopolistic and duopolistic markets. Demand dynamics, cost learning and discounting of future profits are taken into consideration. In addition, both the open-loop and feedback methodologies are pursued to characterize and compare the derived optimal policies.

The paper uses an analytical approach to characterize the optimal dynamic policies in a general setting as is mathematically tractable, followed by the analysis of more specific models to gain additional managerial insights while maintaining a certain degree of generality. Optimal FPS marketing-mix policies are shown to be different from their CDG counterparts for both monopolistic and duopolistic markets. While the ratio of advertising elasticity to price elasticity appears to have been governed by similar set of rules for FPS and CDG, the direction of change of such ratio over time looks different from each other. Managerial implications and directions for future research are also discussed.

1. Introduction

Product and service innovations need to be introduced into their markets with appropriate marketing strategies. The most important marketing variables that affect the dynamic demand of innovations are price and advertising. This paper seeks to qualitatively characterize optimal price, optimal advertising, and the optimal ratio of advertising elasticity of demand to its price elasticity over time. The problem is defined in a continuous-time frame to take advantage of the powerful optimal control methodology (Kamien & Shwartz, 1981). Five major factors influence a firm's optimal marketing-mix strategies and the ratio of advertising elasticity of demand to its price elasticity. They are:

1. Dynamics on the demand side. Demand dynamics refer to all those phenomena that cause the likelihood to buy increase such as innovative behavior, word of mouth, and saturation (Dockner and Jørgensen, 1988a).
2. Dynamics of the supply side. Cost dynamics refer to the cost learning curve: unit costs are assumed to decline with increased accumulated sales. (Clarke, Darrough & Heinke, 1982; Kalish, 1983).
3. Market structure. A monopoly scenario is considered appropriate in the presence of patent protection or by studying a new product/ service in an initial phase before the entry of rivals. After entry of competitors, the market becomes an oligopoly. What basically sets oligopoly apart from monopoly are the strategic interdependence and interactions between rivals (Eliashberg & Jueland, 1986).
4. Multi-period planning. The discount rate r measures how profits earned today are preferred to those earned tomorrow. In the extreme cases $r = 0$ and $r = \infty$ we have far-sighted and myopic agents, respectively. More realistic, but also technically more difficult, are the intermediate cases where $0 < r < \infty$ (Dockner & Jørgensen, 1992).
5. The type of good considered. Frequently purchased products (soap, or toothpaste) are considered similar to subscription

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services (Internet, or heating and air conditioning service contracts) as they face the same dynamics of adoption and/ or retention together with similar revenue rate functions (we are indebted to an anonymous reviewer for bringing the issue to our attention). Consumer durable goods (CDG) purchased for the first time, on the other hand, are treated separately as they are conceived to be different in a mathematical sense from frequently purchased products and services (FPS) since their revenue rate functions are dissimilar resulting in generally divergent marketing-mix strategies. For an FPS good, the revenue rate is equal to the number of subscribers, or sales at time t, N_t, multiplied by the subscription charge, or price P_t whereas it is (dN_t/dt) multiplied by the price P_t for an CDG product, where N_t is the cumulative number of adopters by time t (Libai, Muller & Peres, 2009).

In this paper we analyze both FPS and CDG products in monopolistic as well as duopolistic markets. The paper uses an analytical approach to characterize the optimal dynamic policies in a general setting as is mathematically tractable. This will yield policy conclusions for broad classes of demand and cost functions. As can be imagined, additional assumptions have to be made for our approach to carry through. Nevertheless, a certain degree of generality can be maintained.

Our approach is similar to those related to CDG of Kalish (1983) who considers dynamic pricing in a monopoly, Dockner and Jørgensen (1998a) who consider dynamic advertising in a monopoly, Mesak and Clark (1998) who consider dynamic pricing and advertising in a monopoly, Dockner and Jørgensen (1998b) who consider dynamic pricing competition, and Dockner and Jørgensen (1992) who consider dynamic advertising competition. The findings of the above five studies are able to be affirmed through our research as special cases of our framework. Furthermore, while the above studies arrive at only open-loop strategies, our paper derives both open-loop and feedback strategies. The general structure related to FPS goods reported herein and its comparison to its CDG counterpart for both monopolistic and duopolistic markets are unique to the present study. The paper unifies and generalizes several dynamic models of pricing and advertising in the literature by assuming a general model and by analyzing monopoly and duopoly markets. The main results about the ratio of advertising to price elasticity generalize beyond the classical results. Other results are interesting and/or new. In addition while Mahajan, Muller and Bass (1993) state that a good part of our intuition concerning optimal marketing policies seems to carry over from monopoly to oligopoly, we show in this research that this assertion *may* be the case in the lack of product/ service interdependence, strategic interdependence, or both.

Because the present study is broad in orientation, the dynamic demand models analyzed are assumed *autonomous* (demand does not depend on time explicitly whereas variables are time dependent). Each firm generally controls one pricing instrument and one advertising instrument in the presence of one related state variable. Goodwill models and market share models are excluded from the analysis (readers interested in such models are referred to Huang, Leng & Liang, 2012 for a recent review).

Having positioned the current investigation within the relevant literature and briefly demonstrated the main contributions of our study, we highlight next the structure of the paper. The second section provides a related literature review. The third section outlines a general dynamic model for FPS in a monopoly, formulates the problem, presents the solution method and characterizes the optimal marketing-mix policies. The fourth section sheds light on the same topics depicted in the third section in relation to CDG in a monopoly. The fifth section addresses the similar issues discussed in the third section in FPS duopolistic markets. The sixth section deals with similar issues discussed in the fourth section in CDG duopolistic markets. The last section summarizes and concludes the study. To improve exposition, the derivation of key formulas and proofs of all reported propositions (a total of four) and results (a total of eight) are included in appendices A and B of a separate supplementary component. In addition, a total of eight illustrative examples mostly extracted from the relevant literature are introduced in the main text.

2. Literature review

In this section, we mention a few articles that dealt with models optimized using optimal control theory in relation to frequently purchased products and services (FPS), followed by those related to consumer durable goods (CDG). The first type of models pertains to frequently purchased goods and services (FPS), and the second type of models pertains to consumer durable goods (CDG). Product categories are then broken down into the following specific models: pricing decision models, advertising decision models, and both price and advertising decision models. Monopolistic models are then introduced for each category in the beginning and are followed by competitive models that demonstrate differential games. It is important to note that this review is not intended to be exhaustive. Rather, it aims to enlighten the readers on the significance of the approach undertaken in this article.

2.1. Review of FPS dynamic models

Monopolistic pricing models include Fruchter and Rao (2001), Mesak and Darrat (2002), Schlereth, Skiera and Wolk (2011) and Fruchter and Sigué (2013). Monopolistic advertising models include Sasieni (1971), Sethi (1973, 1975), Hahn and Hyun (1991), Feinberg (2001), and Mesak, Bari, Babin, Birou and Jurkus (2011). An example of monopolistic model that includes both price and advertising is Avagyan, Esteban-Bravo and Vidal-Sanz (2014).

Models of competitive pricing include Chintagunta and Rao (1996) and Chatterjee and Crosbie (1999). Models of advertising competition include Deal (1979), Deal, Sethi and Thompson (1978), Jørgensen (1982), Feichtinger and Dockner (1984), Erickson (1985, 1995a, 1995b, 2009). Examples of competitive models that include both price and advertising are Dockner and Feichtinger (1986) and Chintagunta, Rao and Vilcassim (1993).

2.2. Review of dynamic CDG models

Monopolistic pricing models include Robinson and Lakhani (1975), Bass (1980), Dolan and Jeuland (1981), Bass and Bultez (1982), Clarke et al. (1982), Kalish (1983), Krishnan, Bass and Jain (1999). Monopolistic advertising models include Horsky and Simon (1983), Dockner and Jørgensen (1988a) and Fruchter and Van den Bulte (2011). Examples of monopolistic models that include both price and advertising are Teng and Thompson (1985), Mesak and Clark (1998) and Sethi, Prasad and He (2008).

Models of competitive pricing include Dockner (1985), Wernerfelt (1986), Eliashberg and Jeuland (1986), Dockner and Jørgensen (1988b) and Dockner and Gaunersdorfer (1996). Models of advertising competition include Teng and Thompson (1983), Chintagunta and Vilcassim (1992) and Dockner and Jørgensen (1992). Examples of competitive models that include both price and advertising are Thompson and Teng (1984), Krishnamoorthy, Prasad and Sethi (2010) and Helmes and Schlosser (2015).

For a more comprehensive reviews on FPS and CDG dynamic marketing models analyzed using optimal control theory, interested readers could review survey articles (e.g., Jørgensen, 1982; Feichtinger and Jørgensen, 1983: Mahajan, Muller & Bass, 1990;

Meade & Islam, 2006; Peres, Muller & Mahajan, 2010) and scholarly books (e.g., Dockner, Jørgensen, Van Long & Sorger, 2000; Eliashberg & Lilien, 1993; Erickson, 2003; Haurie, Krawczyk & Zaccour, 2012; Jørgensen & Zaccour, 2004; Mahajan, Muller & Wind, 2000).

In closing this section, we make the following observations based on the reviewed literature:

(1) Analytical tractability still imposes severe limitations in arriving at feedback solutions (optimal trajectories are state dependent) compared to open-loop solutions (optimal trajectories are time dependent determined at the outset of the planning horizon). Therefore, it appears that open-loop solutions are more appealing particularly for short planning horizons. Though the solutions related to these two methodologies are generally different, in some situations they are similar (see e.g., Fershtman, 1987). In this regard and particularly for competitive markets, Jørgensen and Zaccour (2004, p. 10) mention that the main obstacle to the characterization of a feedback equilibrium is the necessity of obtaining the solution of the Hamilton-Jacobi-Bellman (HJB) partial differential equations that can be highly nonlinear, and a general theory of partial differential equations does not exist. However, this article analytically derives in Appendix B feedback strategies for frequently purchased products and services FPS in monopolistic and duopolistic markets.

(2) There is a scarcity of models that incorporate marketing-mix variables of price and advertising at either the monopolistic or the competitive levels. When such models are built, they are not governed by empirically validated theories dictating how to incorporate either or both variables in the dynamic models. To guard against the possibility that the optimal trajectories could be sensitive to the particular functional forms being chosen, our suggested proposal is to employ more flexible (i.e., more general) marketing-mix dynamic models. The above point of view is shared by Dockner and Jørgensen (1988b, p. 319) and Dockner and Jørgensen (1992, p. 460).

(3) The literature indicates that several results arrived at upon analyzing monopolistic markets carry over to competitive settings (e.g., Dockner & Jørgensen, 1988b, p. 319). It is demonstrated in this article that such extension is attributed to the imposition of additional assumptions to gain further managerial insights, without the need to adhere to numerical analyses the outcomes of which could be largely dependent upon the values assigned to model parameters.

3. Frequently purchased products and services FPS in a monopoly

In this section we provide a general model formulation for FPS in a monopoly, followed by a derivation of optimal advertising-mix policies for FPS in a monopoly.

3.1. General model formulation and solution concept for FPS in a monopoly

Let us consider frequently purchased products and services (FPS) in a monopolistic market (in this section as well as in the next section, sales and number of service subscribers are used interchangeably to mean the same thing in a *mathematical sense*). A firm manipulates its price P_t and advertising expenditure U_t (both assumed to be bounded from above) at each time t over a fixed planning period T, $0 \le t \le T$. The monopoly assumption may seem reasonable in situations in which the firm enjoys a patent protection, a proprietary technology, or a dominant market share.

A general demand rate model is given by

$$dN_t/dt = \dot{N}_t = f(N_t, P_t, U_t),\ N_0 \ge 0 \text{ and fixed}, \tag{1}$$

where N_t and $\dot{N}_t$ represent sales (number of subscribers) at time t and the rate of change in sales (subscribers) at t, respectively. Expression (1) suggests that the rate of change in sales is related to current sales and the current rate of the marketing variables. The demand function is *autonomous* as it does not depend on time explicitly. Function f is assumed to be twice differentiable with the following properties related to the marketing variables where a subscript on a function denotes partial differentiation with respect to that subscript:

$$f \ge 0;\ f_P < 0;\ f_U > 0;\ f_{PP} < 0;\ f_{PU} \le 0;\ f_{UU} < 0; \\ \text{and } f_{PP} f_{UU} - f_{pU}^2 > 0. \tag{2}$$

The inequalities (2) imply that sales is non-negative (new customers' acquisition rate is at least equal to customers' defection rate), decreases with an increase in price, and increases with an increase in advertising. Inequality (2) further asserts that price may interact with advertising in affecting the demand rate f and the nature of the interaction is non-positive. A support of this assumption is provided by Kaul and Wittink (1995) in their empirical generalization finding out that (a) an increase in price advertising leads to higher price sensitivity among customers, and (b) the use of price advertising leads to lower prices. The last inequality in 2 together with the propertie $f_{PP} < 0$ *and* $f_{UU} < 0$ imply that the demand function f is concave in the decision variables P and U so that one of the sufficiency conditions of optimality (the Heissian matrix of second partial derivatives of the Hamiltonian H given in (5) is negative definite) is satisfied.

We introduce next a cost learning curve by assuming that marginal costs, denoted by C, depend on the number of subscribers such that marginal costs decrease with increasing the number of subscribers (experience) (Boone, Ganeshan & Hicks, 2008; Chambers & Johnson, 2000),

$$C_t = C(N_t),\ dC(N_t)/dN_t = C_{N_t} = C'(N_t) \le 0. \tag{3}$$

Note that marginal costs could be constant ($C' = 0$). C_t is mainly a function of efforts related to service activation (e.g., installation) and account maintenance (e.g., billing, computer server space, and help provided by the service firm).

For a firm that aims to find the optimum trajectories P_t^* and U_t^* to maximize the discounted profit stream over the planning period T, the problem is formulated as follows for a discount *rate* $r \ge 0$:

$$Max_{P_t, U_t} \int_0^T e^{-rt}[(P_t - C(N_t))N_t - Q(U_t)]dt + e^{-rT} S.N(T), \tag{4}$$

subject to $dN_t/dt = \dot{N}_t = f(N_t, P_t, U_t)$, and the initial number of subscribers $N_0 \ge 0$ is fixed and N_T is free.

In expression (4), $P_t\ N_t$ represents the total revenue generated from subscribers and $C(N_t)\ N_t$ is the related total variable cost. In expression (4), $Q(U_t)$ is the advertising cost function assumed to be non-negative and convex with respect to its argument with the properties $Q' > 0$ *and* $Q'' \ge 0$ (Piconni & Olson, 1979). The term $S\ .\ N(T)$ in (4) is a salvage value which is included to take into account that the time is truncated at $t = T$. A zero salvage value specification ($S = 0$) would be appropriate for an industry characterized by rapid product obsolescence or short product life cycles whereas a specification $S > 0$ would be appropriate for a firm with high brand equity (Raman, 2006). Obviously, for an infinite planning horizon there would be no need to incorporate a salvage value term.

The optimal control problem (4) can be solved by applying Pontryagin's maximum principle optimization technique (Pontryagin, 1962). To apply the maximum principle, we start by forming the current value Hamiltoian (Sethi & Thompson, 2000)

$$H_t(P_t, U_t, N_t) = (P_t - C(N_t))\,N_t - Q(U_t) + \lambda_t f(N_t, P_t, U_t), \tag{5}$$

where λ_t is a costate variable that must satisfy the ensuing requirements:

$$d\lambda_t/dt = r\lambda_t - \partial H_t/\partial N_t, \text{ and the transversality condition } \lambda_T = S. \tag{6}$$

An economic interpretation of λ_t is found in Sethi and Thompson (2000). Briefly, λ_t has the interpretation of a *shadow price* of the stock of subscribers N_t. In this paper, we consider admissible controls that are twice differentiable in t and satisfy $P_t \geq 0$ and $U_t \geq 0$ for all relevant t. (In what follows, the time argument is eliminated as deemed appropriate to minimize confusion and improve clarity). Confining our interest to admissible controls, the partial derivatives of the current value Hamiltonian with respect to P and U along the optimal trajectories, as in Feichtinger (1982), must satisfy the following conditions for an interior solution for which $0 \leq \underline{P} \leq P \leq \bar{P}$ and $0 \leq \underline{U} \leq U \leq \bar{U}$:

$$\partial H/\partial P = 0,\ \partial H/\partial U = 0, \tag{7}$$

where $\underline{P}$ and $\bar{P}$ are the lower and upper bounds of P whereas $\underline{U}$ and $\bar{U}$ are the lower and upper bounds of U, and

Matrix ***HM*** is a *dominant diagonal* negative definite matrix, (8)

such that ***HM*** is a non-singular Hessian matrix of the second partial derivatives of the Hamiltonian H with the properties $|\partial^2H/\partial P^2| > |\partial^2H/\partial P\partial U|$, $|\partial^2H/\partial U^2| > |\partial^2H/\partial U\partial P|$ so that $(\partial^2H/\partial P^2)(\partial^2H/\partial U^2) - (\partial^2H/\partial P\partial U)(\partial^2H/\partial U\partial P) > 0$, and

$$\boldsymbol{HM} = \begin{bmatrix} \partial^2H/\partial P^2 & \partial^2H/\partial P\partial U \\ \partial^2H/\partial U\partial P & \partial^2H/\partial U^2 \end{bmatrix}. \tag{9}$$

It is noted again that the last two inequalities in (2) guarantee that the elements lying on the diagonal of matrix ***HM*** are negative and ***HM*** is negative definite (details are found in expression (A17a) of Appendix A). Conditions (6) and (7) are the necessary conditions of optimality. The last condition (8) is one of several sufficiency conditions of optimality implying that the Hamiltonian H is jointly concave in the control variables P and U together with the state variable N (Seierstad & Sydsaeter, 1977). If a sufficiency condition is violated (e.g., $\partial^2H/\partial P^2 > 0$) the optimal pricing policy would be either the constant $\underline{P}$ or the constant $\bar{P}$. By substituting into the Hamiltonian H, the constant that maximizes H would be chosen as P_t^* (Teng & Thompson, 1985 p. 192).

3.2. Optimal dynamic marketing-mix policies for FPS in a monopoly

This section starts by analyzing the situation of the general FPS model (1) followed by an analysis of one plausible specific model of a generalized mathematical structure.

Using conditions (7) in conjunction with expressions (1), (5) and (6), we derive in the Appendix the contents of the proposition shown below.

Proposition 1. *With demand rate dN/dt given by (1) and the necessary conditions (6) and (7) together with a presence of cost learning, and discounting, then the following relationships hold at any point in time along the optimal trajectories of the marketing-mix variables for FPS providers:*

(i) *The ratio R of the advertising elasticity of the demand rate ($\xi = Uf_U/f$) to its price elasticity ($\Lambda = -Pf_P/f$) equals the ratio of advertising to sales revenue (U/NP), multiplied by the marginal cost of advertising (Q́).*

(ii) *The time derivative of price, dP/dt, and the time derivative of advertising, dU/dt, are governed by the two equations written in a matrix format*

$$\begin{bmatrix} dP/dt \\ dU/dt \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -\partial^2H/\partial U^2 & \partial^2H/\partial U\partial P \\ \partial^2H/\partial P\partial U & -\partial^2H/\partial P^2 \end{bmatrix}\begin{bmatrix} f + N(f_N - f f_{PN}/f_P) - f_P(P - C - NC_N) + r\lambda f_P \\ -Q'(f_N - f f_{UN}/f_U) - f_U(P - C - NC_N) + r\lambda f_U \end{bmatrix}, \tag{10}$$

where $\Delta = (\partial^2 H/\partial P^2)\ (\partial^2H/\partial U^2) - (\partial^2 H/\partial P\partial U)^2 > 0$, given the last condition in (2).

(iii) *The time derivative of the ratio R, d R/ dt, is given by*

$$dR/dt = \Big[-(1/P)\,(dP/dt) + \big(Q''/Q' + 1/U\big)(dU/dt) - f/N\Big]R. \tag{10a}$$

The following observations are gleaned from the novel contents of Proposition 1:

(a) Part (i) of the proposition is in essence the theorem of Dorfman and Steiner (1954) for a monopolist facing a price and advertising-dependent static demand, and a linear advertising cost function, generalized to a dynamic FPS setting. Nerlove and Arrow (1962) also arrived earlier at a similar result in a dynamic setting.

(b) Part (ii) of the proposition implies that it would be *sufficient* that both the price elasticity of the demand rate and the advertising elasticity of the same to decrease with sales (number of subscribers) N for the optimal price to be increasing over time and optimal advertising to be decreasing over time when the discount rate is small ($r = 0$). It is noted here that $\partial\,(-f_p\ P/f)/\ \partial\ N \leq 0$ implies that $f_N - ff_{PN}\ /\ f_p \geq 0$ and $\partial\ (f_U\ U/f)/\ \partial\ N \leq 0$ implies that $f_N - f f_{UN}/f_U \geq 0$. Furthermore, general expressions for the time derivatives $dP/\ dt$ and $dU/\ dt$ pertaining to dynamic univariate diffusion models for new FPS are derived from (16) upon substituting $\partial^2H/\partial P\partial U = \partial^2H/\partial U\partial P = 0$.

(c) Part (iii) of the proposition implies that when observation (b) is applicable, the rate of change in R would be decreasing over time as $dP/dt \geq 0$ and $dU/dt \leq 0$ together with $dN/dt \geq 0$ by the assumptions in (2). For a linear advertising cost function Q, this observation asserts that the FPS provider would decide to set the advertising expenditure as a decreasing percentage of sales revenue over time.

Inspired by the generalized Bass model GBM (Bass, Krishnan & Jain, 1994), found empirically to have superior predictive power to its counterparts, we consider an FPS model that shifts demand dynamics multiplicatively. That is

$$dN/dt = f(N, P, U) = g(N).h(P, U). \tag{11}$$

The demand function (11) is assumed to possess the following properties:

$$g \geq 0;\ h \geq 0;\ h_P < 0;\ h_U > 0;\ h_{PP} < 0;\ h_{PU} \leq 0;\ h_{UU} < 0;$$
$$\text{and } h_{PP}h_{UU} - h_{PU}^2 > 0. \tag{12}$$

The properties (12) are consistent with those in (2) and achieve a similar purpose (Matrix ***HM*** is negative definite).

We are now in a position to introduce the following novel result upon applying (10):

Result 1. For $r = 0$, presence of cost learning curve and a separable demand function for FPS given by (11)

(i) The ratio R of the advertising elasticity of the demand rate ($\xi = Uf_U/f$) to its price elasticity ($\Lambda = -Pf_P/f$) equals the ratio of advertising to sales revenue (U/NP), multiplied by the marginal cost of advertising (Q́).

(ii) Optimal price is increasing over time.

(iii) Optimal advertising is decreasing over time.
(iv) The rate of change of the ratio R of the advertising elasticity of demand ($\xi = Uf_U/f$) to its price elasticity ($\Lambda = -Pf_P/f$) is decreasing over time.

Result 1 implies that the optimal pricing policy is to offer a low price when the FPS is introduced and then to increase the price as sales increases and the particular FPS offering moves through its life cycle. The optimal advertising policy is the exact opposite. We note that each of the two models $f = g(N)\ h_1(P).h_2(U)$ and $f = g(N).(h_1(P) + h_2(U))$ could be cast in 12, where h_1 and h_2 are separable functions in P and U, respectively. The qualitative characterization of the marketing-mix policies over time are therefore similar where for the first model $f_{PU} < 0$ whereas for the second model $f_{PU} = 0$. A specific FPS demand model is briefly reviewed below.

Example 1. Avagyan et al. (2014) in a monopolistic version of their study employ an infinite planning horizon and consider a demand function for consumption goods satisfying properties (12) $dN/dt = \dot{N}_t = [(a + u\,N_t/M)(M - N_t) - k\,N_t]\,W_t(P_t, U_t),\ N_0 = 0$ where $W_t(P_t, U_t)$ conveys the impact of price P_t and advertising expenditure U_t on the growth of sales and is given by $W_t(P_t, U_t) = 1 - m\ (\bar{P} - P_t)^2 + b\ \ln\ (U_t)$, where $a, b, u, m, M > 0$. The quantity $\bar{P} \geq 0$ represents an ideal point price whereas $k \geq 0$ represents the defection rate. The firm's present value of future profits to be maximized is given by $\int_0^\infty e^{-rt}[(Pt - C_t)N_t - U_t]dt$. The authors solve their model numerically. For $r = 0$ and for a separable demand function as the one shown above, the qualitative characterization of the optimal dynamic marketing-mix policy would be consistent with Result 1.

In this section as well as in ensuing sections we apply the maximum principle to characterize *open loop* strategies that depend on time. To characterize *feedback* strategies that depend on the state variable, the Hamilton-Jacobi-Bellman equation is employed (Kamien and Schwartz 1991, p. 261). For an infinite planning horizon and an autonomous separable demand function $f = g(N)\ h_1(P).h_2(U)$, feedback strategies are developed in Appendix B. For $r = 0$, the Dorfman-Steiner theorem alluded to in Proposition 1(i) remains applicable whereas the optimal marketing-mix strategies appear different from those reported in Result 1.

4. On comparing FPS optimal policies to CDG counterparts in monopolistic markets

For new consumer durable goods CDG, the counterpart of (4) takes on the following form for a discount rate $r \geq 0$:

$$\underset{P_t,\,U_t}{\text{Max}} \int_0^T e^{-rt}[(P_t - C(N_t))f(N_t, P_t, U_t) - Q(U_t)]dt + e^{-rT}S.N(T)$$

Subject to

$$\dot{N}_t = f(N_t, P_t, U_t), \text{ and the initial adopters } N_0 \geq 0 \text{ and fixed, } N_T \text{ is free.} \quad (13)$$

The current value Hamiltonian for new consumer durables takes on the form

$$H_t(P_t, U_t, N_t) = (P_t - C(N_t) + \lambda_t)\,f(N_t, P_t, U_t) - Q(U_t), \quad (14)$$

where λ_t is a costate variable that must satisfy (6). For new consumer durables, demand function f is assumed to possess the following main properties:

$$f \geq 0;\ f_P < 0;\ f_U > 0;\ f_{UU} < 0;\ \text{and } f_{PP} < 2\,f_P^2/f. \quad (15)$$

The last two inequalities in (15) guarantee that both $\partial^2H/\partial P^2$ and $\partial^2H/\partial U^2$ would be negative (see expressions (A22) and (A23) in Appendix A). It is also shown in Appendix A that the counterpart of Proposition 1 for FPS takes on the form of Proposition 2 for consumer durable goods (CDG) shown below.

Proposition 2. *With demand rate dN/dt given by (13) and the necessary conditions (6) and (7) together with a presence of cost learning, and discounting, then the following relationships hold at any point in time along the optimal trajectories of the marketing-mix variables for CDG providers:*

(i) *The ratio R of the advertising elasticity of the subscription rate ($\xi = Uf_U/f$) to its price elasticity ($\Lambda = -Pf_P/f$) equals the ratio of advertising to sales revenue (U/ f P), multiplied by the marginal cost of advertising (Q΄).*
(ii) *The time derivative of price, dP/dt, and the time derivative of advertising, dU/dt, are governed by the two equations written in a matrix format:*

$$\begin{bmatrix} dP/dt \\ dU/dt \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -\partial^2H/\partial U^2 & \partial^2H/\partial U\partial P \\ \partial^2H/\partial P\partial U & -\partial^2H/\partial P^2 \end{bmatrix}\begin{bmatrix} -(f/f_P)(f f_{PN} - 2f_P f_N) + r\,\lambda f_P \\ (Q'/f_U)(f f_{UN} - f_U f_N) + r\lambda f_U \end{bmatrix}. \quad (16)$$

where $\Delta = (\partial^2\ H/\partial P^2)\ (\partial^2H/\ \partial U^2) - (\partial^2\ H/\ \partial P\partial U)^2 > 0$.

(iii) *The time derivative of the ratio R, d R/ dt, is given by*

$$dR/dt = [-(1/P + f_P/f)(dP/dt) + \left(Q''/Q' + 1/U - f_U/f\right)(dU/dt) - f_N\big]R. \quad (16a)$$

The following observations are derived from the contents of the above proposition:

(a) Part (i) of Proposition 2 is similar to Part (i) of Proposition 1 implying that the Dorfman-Steiner theorem (1954) of static demand is also generalized to a dynamic CDG setting.
(b) For Part (ii) of the proposition, the time derivative of price, *dP/dt*, along the optimal price trajectory in the study of Kalish (1983, p. 140) has the sign of an expression similar to the first entry of the last column vector in (16). Kalish (1983) studied new product diffusion models that incorporate price alone. The time derivative of advertising, *dU/dt*, along the optimal trajectory in the study of Dockner and Jørgensen (1988a, p. 128) has the sign of an expression similar to the second entry of the last column vector in (16). The above authors studied new product diffusion models that incorporate advertising alone.
(c) General expressions for the time derivatives *dP/ dt* and *dU/ dt* pertaining to dynamic univariate diffusion models for new durable products are derived from (16) upon substituting $\partial^2H/\partial P\partial U = \partial^2H/\partial U\partial P = 0$. It is noted from the Appendix that $\partial^2H/\partial U\partial P = \partial^2H/\partial P\partial U$ has the same sign as $f_U - f\ f_{PU}\ /\ f_P$.

We consider next a separable dynamic CDG model of the following form:

$$f(N, P, U) = g(N).h_1(P).h_2(U), \quad (17)$$

and consistent with (15), functions g, h_1 and h_2 would possess the following properties:

$$g \geq 0;\ h_1 \geq 0;\ h_2 \geq 0;\ h_{1P} < 0;\ h_{2U} > 0;\ h_{1PP} < 2\,h_{1P}^2/h_1;\ \text{and } h_{2UU} < 0. \quad (18)$$

The models articulated by Teng and Thompson (1985, Example 3.4) and Mesak and Clark (1998, Proposition 4) can be cast into (17). We are now in a position to introduce the following result:

Result 2. For $r = 0$, presence of cost learning curve and a separable demand CDG function given by (17)

(i) The ratio R of the advertising elasticity of the demand rate ($\xi = Uf_U/f$) to its price elasticity ($\Lambda = -Pf_P/f$) equals the ratio of advertising to sales revenue ($U/\ f\ P$), multiplied by the marginal cost of advertising ($Q´$).

Table 1
Signs of time derivatives for CDG and FPS: $r = 0$.

Model mathematical structure		Sign of derivatives (CDG)		Sign of derivatives (FPS)	
		dP/dt	dU/dt	dP/dt	dU/dt
M1	$f = (aq + bN)(\bar{N} - N)h$	$\partial f/\partial N$	–	+	–
M2	$f = (a + bN)(\bar{N} - N)hq$	$\partial f/\partial N$	0	+	–
M3	$f = (ah + bN)(\bar{N} - N)q$	$\partial f/\partial N + b(\bar{N} - N)q$	0	+	–
M4	$f = (a + bN)(\bar{N} - N)(h + dq)$	$\partial f/\partial N$	$\partial f/\partial N$	+	–

a is the coefficient of external influence, b is the coefficient of internal influence.
$\bar{N}$ is the market potential, d is a constant coefficient.

(ii) Optimal price has the sign of f_N over time.
(iii) Optimal advertising is constant over time.
(iv) The rate of change of ratio R of the advertising elasticity of demand ($\xi = Uf_U/f$) to its price elasticity ($\Lambda = -Pf_P/f$) has the sign of $-f_N$ over time, provided that $f/f_P + P > 0$.

The findings (i) through (iii) reported in Result 2 pertaining to CDG appear consistent with the earlier findings of Mesak and Clark (1998, Proposition 4), but different from their counterparts reported in Result1 associated with FPS. The condition stated in the novel part (iv) of Result 2 is a sufficient condition, but not a necessary one. It is satisfied for the pricing response function $P^{-\eta}$, $\eta > 1$. We illustrate our findings further through introducing an example.

Example 2. In this example, we consider four alternative Bass-type diffusion models that incorporate marketing-mix variables satisfying observation (b) related to Proposition 1 for FPS. We explore in this example whether the signs of the rate of change of the optimal policy are similar (or dissimilar) to their CDG counterparts in the situation $r = 0$. For that matter, we employ a pricing nonlinear response function $h(P)$ and an advertising nonlinear efficiency function $q(U)$ satisfying the properties:

$$h > 0,\ q > 0;\ h' < 0,\ q' > 0;\ q'' < 0. \tag{19}$$

Whenever a parameter in the Bass model is assumed to depend on one or more of the marketing variables, this parameter is simply multiplied, as appropriate, by one or more of the functions h and q. The findings pertaining to four models are summarized in Table 1.

For diffusion model M1, advertising affects the coefficient of external influence as in the study of Horsky and Simon (1983) whereas price affects the demand rate in a multiplicative fashion as in the study of Robinson and Lakhani (1975). For diffusion Model M2, both price and advertising exert a multiplicative effect on the demand rate as articulated in Sethi et al. (2008). For model M3, price affects the coefficient of external influence as proposed in Parker (1992) and advertising affects multiplicatively the demand rate as suggested in Mesak and Clark (1998). For model M4, price and advertising, combined in an additive structure, affect the demand rate multiplicatively as in the Generalized Bass Model (Bass et al., 1994). The differences between CDG and FPS signs reported in Table1 for the same diffusion models are apparent. This is mainly attributed to the fact that the gross profit rate for CDG is equal to $(P - C)\ dN/dt$ whereas it is equal to $(P - C)\ N$ for FPS.

Example 3. Sethi et al. (2008) consider for a zero marginal production cost the dynamic optimization problem: Find optimal trajectories P^* and U^* to Maximize $\int_0^\infty e^{-rt}\,[P\ dN/dt - U^2]\ dt$

$$\text{s.t.}\ dN/dt = \rho P^{-\eta} U\sqrt{1 - N},\ N(0) = N_0,$$

N is defined as the fraction of the cumulative market captured by time t. The quantities ρ and η are positive constants. The authors only develop optimal feedback strategies using the HJB equation and demonstrate that optimal price is constant over time whereas optimal advertising declines over time and is proportional to $\sqrt{1 - N}$. When the pricing response function $P^{-\eta}$ is replaced with a linear counterpart $1 - \eta P$, the characterization of the optimal feedback strategies remain unchanged.

To arrive at the optimal strategies using the maximum principle and apart from discounting ($r = 0$), Result 2 is applicable so that price would be declining over time ($\partial \sqrt{1 - N} / \partial N < 0$) whereas advertising would be constant over time. The optimal feedback strategies developed by the above authors appear different from the optimal strategies based on the maximum principle.

5. Frequently purchased products and services FPS in a duopoly

In this section, we provide a general model formulation for FPS in a duopoly, followed by a derivation of optimal marketing-mix policies for FPS in a duopoly.

5.1. General model formulation and solution concept for FPS in a duopoly

In a duopoly, each firm i faces a demand function f_i given by

$$dN_i/dt = f_i\big((P_i, U_i);\ (P_j, U_j);\ N_i, N_j\big),\ i, j = 1, 2;\ i \neq j. \tag{20}$$

where the pair (P_i, U_i) represents the decision variables of price and advertising of firm i whereas N_1 and N_2 represent the state variables related to firm 1 and firm 2. The demand function f_i is assumed to be twice differentiable with the following main properties with respect to the marketing-mix variables:

$$f_i \geq 0;\ f_{iPi} < 0;\ f_{iPj} \geq 0;\ f_{iUi} > 0;\ f_{iUj} \leq 0;\ f_{iPiPj} \leq 0;\ f_{iUjUj} \geq 0;\ f_{iPi}f_{jPj} - f_{iPj}f_{jPi} > 0;\ f_{iUiUi}f_{jUjUj} - f_{iUiUj}f_{jUiUj} > 0;\ f_{iPiPi} < 0;\ f_{iPiUi} \leq 0;\ f_{iUiUi} < 0 \text{ and } f_{iPiPi}f_{iUiUi} - f_{iPiUi}^2 > 0. \tag{21}$$

Unless otherwise stated, indices i and j are such that $i, j = 1, 2$ and $i \ddagger j$. In (21), the first subscript on a function stands for a certain competitor. Subsequent subscripts (variables) represent the partial differentiation of the function with respect to such subscript(s) (variable(s)). For example, f_{2P1} stands for $\partial f_2/\partial P_1$ and f_{1P1U1} stands for $\partial^2 f_1 / \partial P_1 \partial U_1$.

Except for the last three assumptions, properties (21) are standard assumptions in oligopoly theory (see for examples Friedman, 1977, 1983 and Thépot, 1983). The inequalities (21) imply that the demand rate of firm i is non-negative, decreases with an increase in price P_i, increases with an increase in price P_j, and increases with an increase in advertising U_i at a decreasing rate. An increase in the advertising of firm j causes the sales of firm i to decrease at an increasing rate ($f_{iUj} \leq 0; f_{iUjUj} \geq 0$). The condition $f_{iPiPj} \leq 0$ states that it is more difficult to increase f_i by reducing P_i when P_j is low than when P_j is high. The condition ($f_{iPi}\ f_{jPj} - f_{iPj}\ f_{jPi} > 0$) means that a firm's price change has a higher impact on its own demand than on the competitor's one. The condition ($f_{iUiUi}f_{jUjUj} - f_{iUiUj}f_{jUiUj} > 0$) also implies that a firm's advertising change has a higher impact on its own demand than on the competitor's one. The last three properties are only confined to FPS asserting that price of a firm may interact with its advertising in affecting its own demand where the nature of the interaction is non-positive and demand of each firm is a concave function in its two marketing-mix variables. The presence of the three assumptions will be shown later to be instrumental in deriving ensuing results and demonstrating that the Hessian matrix of the second partial derivatives of each firm is negative definite.

Each firm i strives to independently determine the optimal pair of its decision variables over time (P_i^*, U_i^*) to maximize the present value of profits for a discount rate r_i given by

$$\pi_i = \int_0^T e^{-r_i t} g_i \, dt + e^{-r_i T} \;\; S_i \, . \, N_i(T) = \int_0^T e^{-r_i t}[(P_i - \; C(N_i))N_i \; - Q_i(U_i)]dt + \; e^{-r_i T} S_i . N_i(T), \tag{22}$$

subject to the system of differential Eq. (20), $N_{i0} \geq 0$ is fixed and N_{iT} is free. In expression (22), $Q_i(U_i)$ is an advertising cost function related to firm i assumed to be non-negative and convex with its argument with the properties $Q_i' > 0$ and $Q_i'' \geq 0$.

The two rivals are assumed to choose their control variables simultaneously. When there is no incentive for either rival to alter his / her control variables, then the choices are said to be in equilibrium. In particular, they are said to be in *Nash equilibrium* if

$$\pi_1((P_1^*, U_1^*); (P_2^*, U_2^*)) \geq \pi_1((P_1, U_1); \; (P_2^*, U_2^*)). \tag{23a}$$

and

$$\pi_2((P_1^*, U_1^*); \; (P_2^*, U_2^*)) \geq \pi_2((P_1^*, U_1^*); \; (P_2, U_2)). \tag{23b}$$

The strategies most employed in the application of the theory of differential games are either *open-loop* or *feedback*. Open–loop strategies are ones for which each player chooses all the control variables' values for each point in time at the outset of the game. Feedback strategies imply that the way of modeling a player's behavior is to suppose that he/she can condition his/ her action at each point of time on the basis of the state of the system at that point.

To arrive at the necessary conditions of optimality from the open-loop Nash equilibrium, one starts by forming the current value Hamiltonian

$$H_i = g_i + \sum_{j=1}^{2} \lambda_{ij} f_j, \, i = 1, 2. \tag{24}$$

Where the λ_{ij}'s are costate variables related to competitor i. With the help of the Hamiltonian (24), one arrives at the following conditions:

$$\partial H_i / \partial P_i = \partial g_i / \partial P_i + \sum_{j=1}^{2} \lambda_{ij} \big(\partial f_j / \partial P_i \big) = 0, \, i = 1, 2. \tag{25}$$

$$\partial H_i / \partial U_i = \partial g_i / \partial U_i + \sum_{j=1}^{2} \lambda_{ij} \big(\partial f_j / \partial U_i \big) = 0, \; i = 1, \; 2. \tag{26}$$

$$d\,\lambda_{ij}/dt = r_i \lambda_{ij} - \partial H_i / \partial N_j = r_i \lambda_{ij} - \partial g_i / \partial N_j - \sum_{j=1}^{2} \lambda_{ij} \big(\partial f_i / \partial N_j \big),$$
$$i = 1, \; 2; \; \lambda_{ij}(T) = S_i \; for \; i = j, \; and \; \lambda_{ij}(T) = 0 \; for \; i \neq j. \tag{27}$$

Eqs. (25), (26), (27) together with the initial values of the state variables N_{io} form the set of necessary conditions which every open-loop Nash solution satisfies.

The (2×2) ***HM*** matrix for a monopoly (9) is replaced by the (4×4) ***HM*** matrix for a duopoly depicted in (28a), that is

$$\boldsymbol{HM} = \begin{bmatrix} \partial^2 H_1 / \partial P_1^2 & \partial^2 H_1 / \partial P_1 \partial U_1 & \partial^2 H_1 / \partial P_1 \partial P_2 & \partial^2 \; H_1 / \partial P_1 \; \partial U_2 \\ \partial^2 H_1 / \partial U_1 \; \partial P_1 & \partial^2 H_1 / \partial U_1^2 & \partial^2 H_1 / \partial U_1 \partial P_2 & \partial^2 \, H_1 / \partial U_1 \; \partial U_2 \\ \partial^2 H_2 / \partial P_2 \partial P_1 & \partial^2 H_2 / \partial P_2 \partial U_1 & \partial^2 H_2 / \partial P_2^2 & \partial^2 \, H_2 / \partial P_2 \; \partial U_2 \\ \partial^2 H_2 / \partial U_2 \partial P_1 & \partial^2 H_2 / \partial U_2 \partial U_1 & \partial^2 H_2 / \partial U_2 \partial P_2 & \partial^2 H_2 / \partial U_2^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{11\,(2x2)} & \boldsymbol{H}_{12\,(2x2)} \\ \boldsymbol{H}_{21\,(2x2)} & \boldsymbol{H}_{22\,(2x2)} \end{bmatrix}. \tag{28a}$$

Among additional sufficient conditions of optimality, matrix ***HM*** in (28a) is assumed to be a dominant negative diagonal definite matrix. It is noted that when the elements of the $\boldsymbol{H}_{12(2x2)}$ and $\boldsymbol{H}_{21(2x2)}$ are zeros, the inverse of the block diagonal matrix ***HM*** is given by

$$\boldsymbol{HM}^{-1} = \begin{bmatrix} {\boldsymbol{H}_{11\,(2x2)}}^{-1} & 0_{(2x2)} \\ 0_{(2x2)} & {\boldsymbol{H}_{22\,(2x2)}}^{-1} \end{bmatrix}. \tag{28b}$$

5.2. Optimal dynamic marketing-mix policies for FPS in a duopoly

This section starts by analyzing the situation of the general FPS model (20) followed by an analysis of two plausible specific models of generalized mathematical structures.

Using conditions (25) and (26) in conjunction with expressions (20), (24) and (27), we derive in Appendix A the contents of the novel proposition shown below.

Proposition 3. *For demand rate dN_i /dt given by (20) and the necessary conditions (25), (26) and (27) together with presence of cost learning, and discounting, then the following relationships hold at any point in time along the optimal trajectories of the marketing-mix variables for FPS providers:*

(i) *The ratio R_i of the advertising elasticity of demand of firm i ($\xi_i = U_i f_{iUi} / f_i$) to its price elasticity ($\Lambda_i = -P_i f_{ipi} / f_i$) equals$(Q_i' \; - \; \lambda_{ij} \; f_{jUi} \;) U_i \; / \; (N_i + \; \lambda_{ij} f_{jPi}) P_i$.*

(ii) *The time derivative of price, dP_i /dt, and the time derivative of advertising, dU_i /dt, i = 1, 2 are governed by the four equations written in a matrix format*

$$\begin{bmatrix} d \; P_1/dt \\ dU_1/dt \\ dP_2/dt \\ dU_2/dt \end{bmatrix} = -\begin{bmatrix} \boldsymbol{H}_{11\,(2x2)} & \boldsymbol{H}_{12\,(2x2)} \\ \boldsymbol{H}_{21\,(2x2)} & \boldsymbol{H}_{22\,(2x2)} \end{bmatrix}^{-1} . \begin{bmatrix} f_1 - \lambda_{11}(f_{1P1} f_{1N1} - f_1 f_{1P1N1} \; + f_{2P1} \; f_{1N2} \; - f_2 \; f_{1P1N2}) - f_{1P1} \; (P_1 - C_1 - N_1 C_{1N1}) \\ + f_{1P1} r_1 \lambda_{11} - \lambda_{12}(f_{1P1} f_{2N1} - f_1 f_{2P1N1} \; + f_{2P1} \; f_{2N2} \; - \; f_2 \; f_{2P1N2}) + f_{2P1} r_1 \lambda_{12}. \\ -\lambda_{11}(f_{1U1} f_{1N1} - f_1 f_{1U1N1} \; + f_{2U1} \; f_{1N2} \; - \; f_2 \; f_{1U1N2}) - \; f_{1U1} \; (P_1 - C_1 - N_1 C_{1N1}) \\ + f_{1U1} r_1 \lambda_{11} - \lambda_{12}(f_{1U1} f_{2N1} - f_1 f_{2U1N1} \; + f_{2U1} \; f_{2N2} \; - \; f_2 \; f_{2U1N2}) + f_{2U1} r_1 \lambda_{12}. \\ f_2 - \lambda_{22}(f_{2P2} f_{2N2} - f_2 f_{2P2N2} \; + f_{1P2} \; f_{2N1} - f_1 \; f_{2P2N1}) - f_{2P2} \; (P_2 - C_2 - N_2 C_{2N2}) \\ + f_{2P2} r_2 \lambda_{22} - \lambda_{21}(f_{2P2} f_{1N2} - f_2 f_{1P2N2} \; + f_{1P2} \; f_{1N1} \; - \; f_1 \; f_{1P2N1}) + f_{1P2} r_2 \lambda_{21}. \\ -\lambda_{22}(f_{2U2} f_{2N2} - f_2 f_{2U2N2} \; + f_{1U2} \; f_{2N1} - \; f_1 \; f_{2U2N1}) - \; f_{2U2} \; (P_2 - C_2 - N_2 C_{2N2}) \\ + f_{2U2} r_2 \lambda_{22} - \lambda_{21}(f_{2U2} f_{1N2} - f_2 f_{1U2N2} \; + f_{1U2} \; f_{1N1} - \; f_1 \; f_{1U2N1}) + f_{1U2} r_2 \lambda_{21}. \end{bmatrix}. \tag{29}$$

(iii) *The time derivative of the ratio R_i, d R_i / dt, is governed by the expression*

$$d\,R_i/dt = (\partial R_i/\partial P_i)(d\; P_i/dt) + \big(\partial R_i/\partial P_j\big)(d\; P_j/dt) + (\partial\, R_i/\partial U_i)(d\; U_i/dt) + \big(\partial R_i/\partial U_j\big)(d\; U_j/dt) + (\partial R_i/\partial N_i) f_i + \big(\partial R_i/\partial N_j\big) f_j + (\partial R_i/\partial \lambda_{ij})(d\lambda_{\;ij}/dt), \; i, \;\; j = 1, 2 \, and \, i \neq j. \tag{29a}$$

The following observations are gleaned from the contents of Proposition 3:

(a) Part (i) of the above proposition for a duopoly is a modified version of the Dorfman - Steiner theorem depicted in par (i) of Proposition 1 for a monopoly. However, it becomes a competitive generalization of it when the mixed state variables $\lambda_{ij} = 0$, or $f_{jPi} = f_{jUi} = 0$, $i \ddagger j$.

(b) For FPS pricing completion only, it can be easily shown that (29) reduces to the form

$$\begin{bmatrix} d\ P_1/dt \\ d\ P_2/dt \end{bmatrix} = -\boldsymbol{HM}^{-1} \begin{bmatrix} f_1 - \lambda_{11}(f_{1P1}f_{1N1} - f_1 f_{1P1N1} + f_{2P1}\ f_{1N2} - f_2\ f_{1P1N2}) - f_{1P1}\ (P_1 - C_1 - N_1 C_{1N1}) \\ +f_{1P1}r_1\lambda_{11} - \lambda_{12}(f_{1P1}f_{2N1} - f_1 f_{2P1N1} + f_{2P1}\ f_{2N2} - f_2\ f_{2P1N2}) + f_{2P1}r_1\lambda_{12}. \\ f_2 - \lambda_{22}(f_{2P2}f_{2N2} - f_2 f_{2P2N2} + f_{1P2}\ f_{2N1} - f_1\ f_{2P2N1}) - f_{2P2}\ (P_2 - C_2 - N_2 C_{2N2}) \\ +f_{2P2}r_2\lambda_{22} - \lambda_{21}(f_{2P2}f_{1N2} - f_2 f_{1P2N2} + f_{1P2}\ f_{1N1} - f_1\ f_{1P2N1}) + f_{1P2}r_2\lambda_{21}. \end{bmatrix}, \quad (29b)$$

and $\boldsymbol{HM} = \begin{bmatrix} \partial^2 H_1/\partial P_1^2 & \partial^2 H_1/\partial P_1 \partial P_2 \\ \partial^2 H_2/\partial P_2 \partial P_1 & \partial^2 H_2/\partial P_2^2 \end{bmatrix}$.

(c) For FPS advertising competition only, it can be easily shown that (29) reduces to the form

$$\begin{bmatrix} d\ U_1/dt \\ d\ U_2/dt \end{bmatrix} = -\boldsymbol{HM}^{-1} \begin{bmatrix} -\lambda_{11}(f_{1U1}f_{1N1} - f_1 f_{1U1N1} + f_{2U1}\ f_{1N2} - f_2\ f_{1U1N2}) - f_{1U1}\ (P_1 - C_1 - N_1 C_{1N1}) \\ +f_{1U1}r_1\lambda_{11} - \lambda_{12}(f_{1U1}f_{2N1} - f_1 f_{2U1N1} + f_{2U1}\ f_{2N2} - f_2\ f_{2U1N2}) + f_{2U1}r_1\lambda_{12}. \\ -\lambda_{22}(f_{2U2}f_{2N2} - f_2 f_{2U2N2} + f_{1U2}\ f_{2N1} - f_1\ f_{2U2N1}) - f_{2U2}\ (P_2 - C_2 - N_2 C_{2N2}) \\ +f_{2U2}r_2\lambda_{22} - \lambda_{21}(f_{2U2}f_{1N2} - f_2 f_{1U2N2} + f_{1U2}\ f_{1N1} - f_1\ f_{1U2N1}) + f_{1U2}r_2\lambda_{21}. \end{bmatrix}, \quad (29c)$$

and $\boldsymbol{HM} = \begin{bmatrix} \partial^2 H_1/\partial U_1^2 & \partial^2 H_1/\partial U_1 \partial U_2 \\ \partial^2 H_2/\partial U_2 \partial U_1 & \partial^2 H_2/\partial U_2^2 \end{bmatrix}$.

(d) An expanded expression of part (iii) of the proposition is found in Appendix A.

(e) As can be imagined from the contents of parts (ii) and (iii) of the proposition, additional assumptions have to be made to gain managerial insights. Nevertheless, a certain degree of generality can be maintained.

In the next two subsections two alternative scenarios pertaining to the demand function (20) are examined.

5.2.1. Competition with a firm's adoption effect only and marketing mix-variables for all

For this scenario, the demand functions for two rivals are as shown below

$$dN_1/dt = f_1(N_1, N_2, P_1, U_1, P_2, U_2) = f_1(N_1, P_1, U_1, P_2, U_2). \quad (30a)$$

$$dN_2/dt = f_2(N_1, N_2, P_1, U_1, P_2, U_2) = f_2(N_2, P_1, U_1, P_2, U_2). \quad (30b)$$

This case describes a situation where firms base their decisions on market segment dynamics. This may be a reasonable hypothesis in a case where there is low substitutability between the products/services (Dockner and Jørgensen (1992, p. 468). In addition to the assumptions (21), the following additional assumptions are being made:

$$f_{iPiUj} = f_{iPiPj} = f_{iUiPj} = f_{iUiUj} = 0, \quad i \neq j, \text{ and} \quad (31)$$

$$f_i + N_i(f_{iNi} - f_i\ f_{iPiNi}/f_{iPi}) \geq 0, \ (f_{iNi} - f_i f_{iUiNi}/f_{iUi}) \geq 0 \ for\ i = 1, 2. \quad (31a)$$

The additional assumptions in (31) imply that each of the demand functions is *additively* separable with respect to the marketing-mix variables pair (P_ℓ, U_ℓ), $\ell = i, j$ so that the Hessian matrix $\boldsymbol{HM}$ takes on the form depicted in (28a) of an inverse given by (28b). For the demand scenario considered in this subsection, it is shown in the Appendix that $\boldsymbol{\lambda}_{11} > 0$ and $\boldsymbol{\lambda}_{22} > 0$ whereas $\boldsymbol{\lambda}_{12} = \boldsymbol{\lambda}_{21} = 0$. Conditions (31a) are shown to be instrumental in arriving at Results 3 and 4. As mentioned in observation (b) related to Proposition 1, assumptions (31a) are satisfied when both the price elasticity of the demand rate of each firm i and the advertising elasticity of the same decrease with number of subscribers (sales) N_i We are now in a position to introduce our findings shown in novel Result 3.

Result 3. For $r_i = 0$, presence of cost learning curve and FPS demand functions given by $dN_i/dt = f_i(N_i, P_1, U_1, P_2, U_2)$ satisfying properties (21) and the additional properties (31) together with the sufficient conditions (31a) for all i

(i) The ratio R_i of the advertising elasticity of demand of firm i $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ equals the ratio of advertising to sales revenue of firm i $(U_i/N_i P_i)$, multiplied by Q_i'.

(ii) Optimal price is increasing over time for each competitor i.

(iii) Optimal advertising is decreasing over time for each competitor i.

(iv) The ratio of the advertising elasticity of demand $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ is decreasing over time for each competitor i.

Result 3 generalizes the Dorfman-Steiner theorem alluded to in Proposition 1(i) and Result 1 for a dynamic monopoly to a FPS dynamic duopoly. In short, under demand scenario 5.2.1 for FPS and the additional assumptions (31) and (31a), monopoly findings shown in Result 1 carry over in their entirety to a duopoly.

Example 4. Jørgensen (1982) considers a differential game of excess advertising for two service providers (sellers) where subscribers (buyers) are perfectly mobile and switch instantly to the firm which has the larger rate of advertising expenditure. The differential game is summarized as follows:

Find optimal trajectories U_i^* to maximize $\int_0^T e^{-r_i t}(q_i\ N_i\ - U_i)\ dt$, $N_i(0) = N_{i0} \geq 0$, s.t. $dN_i/\ dt = f_i = k \ln (U_i/\ U_j) = k\ (\ln U_i - \ln U_j)$, $i = j = 1, 2;\ i \neq j$, where k is a positive constant, q_i is a constant unit profit margin, and $N_1 + N_2 = M$, M is a positive market potential.

For zero discounting $(r_i = 0)$, the above author derives optimal open-loop advertising strategies that are decreasing over time for both firms. The above findings are readily available from (29c) as for the considered demand functions $\partial^2 H_1/\partial U_1 \partial U_2 = \partial^2 H_2/\partial U_2^2 = 0$ and the first entry of the last column vector located at the right-hand-side of (29c) is $[-\ f_{1U1}\ (P_1 - C_1 - N_1 C_{1N1}) < 0]$ and also its second entry $[-\ f_{2U2}\ (P_2 - C_2 - N_2 C_{2N2}) < 0]$. It is further noted from (29c) that the findings of Jørgensen (1982) are extendable to the case for which cost learning/economics of scale are present.

5.2.2. Competition with marketing-mix variables and adoption for all

For this scenario, the demand functions for two rivals are as shown below.

$$dN_1/dt = f_1(N_1, N_2, P_1, U_1, P_2, U_2) = f_1(N_1, N_2, P_1, U_1) \quad (32a)$$

$$dN_2/dt = f_2(N_1, N_2, P_1, U_1, P_2, U_2) = f_2(N_1, N_2, P_2, U_2). \quad (32b)$$

The demand functions (32) mean that the sales of each rival only depend on his/ her marketing mix variables and on the sales (subscriptions) of both firms. Models of similar structures as (32) but for only one marketing variable has appeared in the literature (e.g., Erickson, 2009). In addition to satisfying the assumptions (21), f_1 and f_2 are assumed to satisfy the assumptions

$$f_{1N2} \leq 0, \text{ and } f_{2N1} \leq 0. \quad (33)$$

By the construction of the demand functions of both firms, $\boldsymbol{\lambda}_{11} > 0$ and $\boldsymbol{\lambda}_{22} > 0$ whereas the Hessian matrix $\boldsymbol{HM}$ takes on the form depicted in (28a) of an inverse given by (28b). Such additional assumptions (33) assert that an increase in the number of subscribers of a firm would decrease the future demand of its rival (see Examples 5 and 6 below for plausible illustrations).

Such assumptions imply that $\boldsymbol{\lambda}_{12} < 0$ and $\boldsymbol{\lambda}_{21} < 0$. We are now in a position to introduce our findings shown in novel Result 4.

Result 4. For $r_i = 0$, presence of cost learning curve and FPS demand function of firm i given by $dN_i/dt = f_i(N_i, P_1, U_1, P_2, U_2)$ satisfying properties (21) and the additional properties (33) together with the *sufficient* conditions (31a) for $i = 1, 2$.

(i) The ratio R_i of the advertising elasticity of demand of firm i $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ equals the ratio of advertising to sales revenue of firm i $(U_i/N_i P_i)$, multiplied by Q_i'
(ii) Optimal price is increasing over time for each competitor i.
(iii) Optimal advertising is decreasing over time for each competitor i.
(iv) The ratio of the advertising elasticity of demand $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ is decreasing over time for each competitor i.

Result 4 also generalizes Proposition 1(i) and Result 1 for a dynamic monopoly to a dynamic duopoly. In short, under demand scenario 5.2.2 for FPS and the additional assumptions (33), monopoly finding shown in Result 1 carry over in their entirety to a duopoly. The contents of Result 4 are also consistent with those related to Result 3.

For an infinite planning horizon and an autonomous separable demand function $f_i = g(N_1, N_2)\, h_i(P_i).h_j(P_j).q_i(U_i).q_j(P_j)$, feedback strategies are developed in Appendix B. For $r_i = 0$, the Dorfman - Steiner theorem alluded to in Results 3(i) and 4(i) remains applicable whereas the optimal marketing-mix strategies appear different from those reported in Results 3 and 4.

Example 5. Erickson (2009) considers a differential game of advertising completion summarized as follows:

Find optimal trajectory U_i^* to Maximize $\int_0^\infty e^{-r_i t}(q_i N_i - U_i^2)\, dt$, s.t. $d N_i/dt = \beta_i U_i \sqrt{M - N_1 - N_2} - \rho_i N_i$, $N_i(0) = N_{i0}$, $i = 1,2$,

where N_1 and N_2 are the sales of Firm *1* and Firm *2* respectively at time t. The quantities M, q_i, β_i and ρ_i are positive constants. The above author only develops feedback strategies and demonstrates that for a symmetric completion ($\beta_i = \beta$, $\rho_i = \rho$ and $q_i = q$), optimal advertising policies for both firms are decreasing over time and each is proportional to $\sqrt{M - N_1 - N_2}$. To arrive at optimal open-loop advertising policies for the situation of no discounting ($r_i = 0$), we apply (29c) and Result 4 to produce optimal advertising policies that are decreasing over time, provided that $N_i \le M/3$ as a *sufficient* condition. Furthermore, for small defection rates ($\rho_i = 0$), the open loop advertising policies would be monotonically decreasing over time. It is further noted that for the above Erickson's model, the additional assumptions (33) are met.

Example 6. Inspired by the work of Feinberg (2001), consider a game of a marketing-mix competition summarized as follows:

Find optimal trajectories P_i^* and U_i^* to Maximize $\int_0^\infty e^{-r_i t}[(P_i - C_i)N_i - Q_i)\, dt$, s.t. $d N_i/dt = h_{ii}(P_i, U_i) N_i (M - N_1 - N_2) - \rho_i \sqrt{N_i}$, $N_i(0) = N_{i0}$, ρ_i *and* M *are parameters;* $i = 1,2$.

To arrive at open loop marketing-mix policies for the situation of no discounting ($r_i = 0$), we apply (29) and Result 4 to produce optimal pricing policy that is increasing over time and advertising policy that is decreasing over time, provided that $3N_1 + N_2 \le M$, $N_1 + 3N_2 \le M$, $N_1 + N_2 \le M$, $N_1 \ge 0$, $N_2 \ge 0$ as *sufficient* conditions. Furthermore, for small defection rates ($\rho_i = 0$), the open-loop policies would be as described in Result 4 [additional sufficient conditions are not needed]. It is further noted that for the above model specification, the assumptions (33) are satisfied.

6. Optimal policies of CDG in duopolistic markets

For new consumer durable goods CDG, the counterpart of (22) takes on the form

$$\pi_i = \int_0^T e^{-r_i t} g_i\, dt + e^{-r_i T} S_i N_i(T) = \int_0^T e^{-r_i t}\big[(P_i - C(N_i))\dot{N}_i - Q_i(U_i)\big]dt + e^{-r_i T} S_i N_i(T). \tag{34}$$

Each firm i strives to independently determine the optimal pair of its decision variables over time (P_i^*, U_i^*) to maximize the present value of profits (34) subject to the system of differential Eqs. (20), $N_{i0} \ge 0$ is fixed and N_{iT} is free. In expression (34), $Q_i(U_i)$ is an advertising cost function related to firm i assumed to be non-negative and convex with respect to its argument of properties $Q_i' > 0$ and $Q_i'' \ge 0$.

The current value Hamiltonian for new consumer durables takes on the form

$$H_i = [(P_i - C(N_i))(dN_i/dt) - Q_i(U_i)] + \sum_{j=1}^{2} \lambda_{ij} f_j,\ i = 1, 2; \tag{35}$$

where the λ_{ij}'s are costate variables related to competitor i that must satisfy (27). For new consumer durables, demand function f_i is assumed to possess the following properties:

$$f_i \ge 0;\ f_{iPi} < 0;\ f_{iPj} \ge 0;\ f_{iUi} > 0;\ f_{iUj} \le 0;\ f_{iPiPj} \le 0;\ f_{iUiUi} \le 0;\ f_{iUjUj} \ge 0;\ f_{iPi} f_{jPj} - f_{iPj} f_{jPi} > 0;\ f_{iUiUi} f_{jUjUj} - f_{iUiUj} f_{jUiUj} > 0;\ \text{and } f_{iPiPi} < 2 f_{iPi}^2 / f_i. \tag{36}$$

Apart for the last assumption, properties (36) are standard assumptions in oligopoly theory and are interpreted in the same manner as their similar counterparts in (21). The presence of the last assumption in (36) will be shown later to be instrumental in deriving ensuing results and guaranteeing that $\partial^2 H_i/\partial P_i^2$ would be negative.

Using conditions (25) and (26) in conjunction with expressions (20), (35) and (27), we derive in the Appendix the contents of the proposition shown below.

Proposition 4. *For demand functions* dN_i/dt *given by (20) and the necessary conditions (25), (26) and (27) together with presence of cost learning, and discounting, then the following relationships hold at any point in time along the optimal trajectories of the marketing-mix variables for CDG providers:*

(i) *The ratio* R_i *of the advertising elasticity of demand of firm* i $(\xi_i = U_i f_{iUi}/f_i)$ *to its price elasticity* $(\Lambda_i = -P_i f_{ipi}/f_i)$ *equals* $(Q_i' - \lambda_{ij} f_{jUi})U_i / (f_i + \lambda_{ij} f_{jPi})P_i$.
(ii) *The time derivative of price,* dP_i/dt*, and the time derivative of advertising,* dU_i/dt*, are governed by the four equations written in a matrix format*

$$\begin{bmatrix} dP_1/dt \\ dU_1/dt \\ dP_2/dt \\ dU_2/dt \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{11(2x2)} & \boldsymbol{H}_{12(2x2)} \\ \boldsymbol{H}_{21(2x2)} & \boldsymbol{H}_{22(2x2)} \end{bmatrix}^{-1} \begin{bmatrix} (P_1 - C_1 + \lambda_{11})\{f_{1P1N1} f_1 - f_{1P1} f_{1N1} + f_{1P1N2} f_2 - f_{2P1} f_{1N2}\} + \lambda_{12}\{f_{2P1N1} f_1 \\ - f_{1P1} f_{2N1} + f_{2P1N2} f_2 - f_{2P1} f_{2N2}\} + f_{1N1} f_1 + f_{1N2} f_2 + r_1(\lambda_{11} f_{1P1} + \lambda_{12} f_{2P1}). \\ (P_1 - C_1 + \lambda_{11})\{f_{1U1N1} f_1 - f_{1U1} f_{1N1} + f_{1U1N2} f_2 - f_{2U1} f_{1N2}\} + \lambda_{12}\{f_{2U1N1} f_1 \\ - f_{1U1} f_{2N1} + f_{2U1N2} f_2 - f_{2U1} f_{2N2}\} + r_1(\lambda_{11} f_{1U1} + \lambda_{12} f_{2U1}). \\ (P_2 - C_2 + \lambda_{22})\{f_{2P2N2} f_2 - f_{2P2} f_{2N2} + f_{2P1N1} f_1 - f_{1P2} f_{2N1}\} + \lambda_{21}\{f_{1P2N2} f_2 \\ - f_{2P2} f_{1N2} + f_{1P2N1} f_1 - f_{1P2} f_{1N1}\} + f_{2N2} f_2 + f_{2N1} f_1 + r_2(\lambda_{22} f_{2P2} + \lambda_{21} f_{1P2}). \\ (P_2 - C_2 + \lambda_{22})\{f_{2U2N2} f_2 - f_{2U2} f_{2N2} + f_{2U2N1} f_1 - f_{1U2} f_{2N1}\} + \lambda_{21}\{f_{1U2N1} f_2 \\ - f_{2U2} f_{1N2} + f_{1U2N1} f_1 - f_{1U1} f_{1N1}\} + r_2(\lambda_{22} f_{2U2} + \lambda_{21} f_{1U2}). \end{bmatrix}. \tag{37}$$

(iii) *The time derivative of the ratio* R_i*,* $d R_i / dt$*, is governed by the expression*

$$
\begin{aligned}
d R_i/dt &= (\partial R_i/\partial P_i)(d\ P_i/dt) + \left(\partial R_i/\partial P_j\right)(d\ P_j/dt) \\
&+ (\partial R_i/\partial U_i)(d\ U_i/dt) + \left(\partial R_i/\partial U_j\right)(d\ U_j/dt) \\
&+ (\partial R_i/\partial N_i) f_i + \left(\partial R_i/\partial N_j\right) f_j + (\partial R_i/\partial \lambda_{ij})(d\lambda_{ij}/dt).
\end{aligned} \tag{37a}
$$

The following observations are gleaned from the contents of Proposition 4:

(a) Part (i) of the above proposition for a duopoly is a modified version of the Dorfman - Steiner theorem depicted in part (i) of Proposition 2 for a monopoly. However, it becomes a competitive generalization of it when the mixed state variables $\lambda_{ij} = 0$, or $f_{jPi} = f_{jUi} = 0_i$, $i \ddagger j$.

(b) For CDG pricing completion only, it can be easily shown that (37) reduces to the form

$$
\begin{bmatrix} d\,P_1/dt \\ d\,P_2/dt \end{bmatrix} =
-\boldsymbol{HM}^{-1}
\begin{bmatrix}
(P_1 - C_1 + \lambda_{11})\{f_{1P1N1} f_1 - f_{1P1}\ f_{1N1} + f_{1P1N2}\ f_2 - f_{2P1} f_{1N2}\} + \lambda_{12}\ \{f_{2P1N1}\ f_1 \\
- f_{1P1} f_{2N1} + f_{2P1N2}\ f_2 - f_{2P1}\ f_{2N2}\} + f_{1N1} f_1 + f_{1N2} f_2 + r_1(\lambda_{11} f_{1P1} + \lambda_{12} f_{2P1}). \\
(P_2 - C_2 + \lambda_{22})\{f_{2P2N2} f_2 - f_{2P2}\ f_{2N2} + f_{2P1N1}\ f_1 - f_{1P2} f_{2N1}\} + \lambda_{21}\ \{f_{1P2N2}\ f_2 \\
- f_{2P2} f_{1N2} + f_{1P2N1}\ f_1 - f_{1P2}\ f_{1N1}\} + f_{2N2} f_2 + f_{2N1} f_1 + r_2(\lambda_{22} f_{2P2} + \lambda_{21} f_{1P2}).
\end{bmatrix},
$$
$$
\text{and } \boldsymbol{HM} = \begin{bmatrix} \partial^2 H_1/\partial P_1^2 & \partial^2 H_1/\partial P_1 \partial P_2 \\ \partial^2 H_2/\partial P_2 \partial P_1 & \partial^2 H_2/\partial P_2^2 \end{bmatrix}. \tag{37b}
$$

(c) For CDG advertising competition only, it can be easily shown that (37) reduces to the form

$$
\begin{bmatrix} d\ U_1/dt \\ d\ U_2/dt \end{bmatrix} =
-\boldsymbol{HM}^{-1}
\begin{bmatrix}
(P_1 - C_1 + \lambda_{11})\{f_{1U1N1} f_1 - f_{1U1}\ f_{1N1} + f_{1U1N2} f_2 - f_{2U1} f_{1N2}\} + \lambda_{12}\{f_{2U1N1}\ f_1 \\
- f_{1U1} f_{2N1} + f_{2U1N2}\ f_2 - f_{2U1}\ f_{2N2}\} + r_1(\lambda_{11} f_{1U1} + \lambda_{12} f_{2U1}). \\
(P_2 - C_2 + \lambda_{22})\{f_{2U2N2} f_2 - f_{2U2}\ f_{2N2} + f_{2U2N1} f_1 - f_{1U2} f_{2N1}\} + \lambda_{21}\ \{f_{1U2N1}\ f_2 \\
- f_{2U2} f_{1N2} + f_{1U2N1}\ f_1 - f_{1U1}\ f_{1N1}\} + r_2(\lambda_{22} f_{2U2} + \lambda_{21} f_{1U2}).
\end{bmatrix},
$$
$$
\text{and } \boldsymbol{HM} = \begin{bmatrix} \partial^2 H_1/\partial U_1^2 & \partial^2 H_1/\partial U_1 \partial U_2 \\ \partial^2 H_2/\partial U_2 \partial U_1 & \partial^2 H_2/\partial U_2^2 \end{bmatrix}. \tag{37c}
$$

(d) An expanded expression of part (iii) of the proposition is found in Appendix A.

(e) As can be imagined from the abstract contents of parts (ii) and (iii) of the proposition, additional assumptions have to be made to gain managerial insights. Nevertheless, a certain degree of generality can be maintained.

In the next two subsections two alternative scenarios pertaining to the demand function (20) are examined.

6.1. Competition with a firm's adoption effect only and marketing mix-variables for all

The demand functions for two rivals are assumed *multiplicatively* separable as shown below

$$
\begin{aligned}
dN_1/dt &= f_1(N_1, P_1, U_1, P_2, U_2) \\
&= g_1(N_1) h_{11}(P_1).\, h_{12}(P_2).q_{11}(U_1).q_{12}(U_2).
\end{aligned} \tag{38a}
$$

$$
\begin{aligned}
dN_2/dt &= f_2(N_2, P_1, U_1, P_2, U_2) \\
&= g_2(N_2).h_{21}(P_1).h_{22}(P_2).q_{21}(U_1).q_{22}(U_2)
\end{aligned} \tag{38b}
$$

The demand functions (38) mean that the interaction between prices, advertising and experience for each firm is seperably multiplicative. The specific formulation in (38) is inspired by models included in Dockner and Jørgensen (1988b) for pricing competition and Dockner and Jørgensen (1992) for advertising competition. For the demand scenario considered in this subsection, it is shown in Appendix A that $\lambda_{11} > 0$ and $\lambda_{22} > 0$ whereas $\lambda_{12} = \lambda_{21} = 0$. The construction of the demand functions (38) implies that the Hessian matrix ***HM*** takes on the form depicted in (28a) of an inverse given by (28b). We are now in a position to introduce our findings shown in Result 5.

Result 5. For $r_i = 0$, presence of cost learning curve and CDG demand functions given by (38) satisfying properties (36)

(i) The ratio R_i of the advertising elasticity of demand of firm i $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ equals the ratio of advertising to sales revenue of firm i $(U_i / f_i\ P_i)$, multiplied by Q_i'.
(ii) Optimal price has the sign of f_{iNi} for each competitor i.
(iii) Optimal advertising is constant over time for each competitor i.
(iv) The ratio of the advertising elasticity of demand $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ has the sign of $-f_{iNi}$ over time for each competitor i, provided that $f_i/f_{iPi} + P_i > 0$.

Again, the findings reported in Result 5 pertaining to CDG appear different from their counterparts reported in Result 3 associated with FPS, except for part (i) reported in both Results 3 and 5. Result 5 generalizes Proposition 2(i) and Result 2 for a dynamic monopoly to a dynamic duopoly. Furthermore, the findings reported in Dockner and Jørgensen (1988, Theorem 2) are consistent with Result 5 (ii) whereas the findings reported in Dockner and Jørgensen (1992, Theorem 2) are consistent with Result 5(iii). Result 5 (iv) is novel to the literature.

6.2. Competition with marketing-mix variables and adoption for all

For this scenario, the demand functions for two rivals are assumed separable as shown below.

$$
dN_1/dt = f_1(N_1, N_2, P_1, U_1, P_2, U_2) = g_1(N_1, N_2).h_{11}(P_1).q_{11}(U_1). \tag{39a}
$$

$$
dN_2/dt = f_2(N_1, N_2, P_1, U_1, P_2, U_2) = g_2(N_1, N_2).h_{22}(P_2).q_{22}U_2). \tag{39b}
$$

The demand functions (39) mean that sales of each rival depend on his/her marketing-mix variables, but multiplicatively on cumulative sales of both firms. The specific formulation (39) is inspired by the earlier work of Thompson and Teng (1984). In addition to satisfying the assumptions (36), f_1 and f_2 are assumed to satisfy the additional assumptions

$$
f_{1N2} \le 0, \text{ and } f_{2N1} \le 0,\ f_{1N1} \ge 0, \text{ and } f_{2N2} \ge 0. \tag{40}
$$

By the construction of the demand functions of both firms, $\boldsymbol{\lambda}_{11} > 0$ and $\boldsymbol{\lambda}_{22} > 0$ whereas the Hessian matrix ***HM*** takes on the form depicted in (28a) of an inverse given by (28b). The additional assumptions $f_{1N2} \le 0$, and $f_{2N1} \le 0$ in (40) imply that $\boldsymbol{\lambda}_{12} < 0$ and $\boldsymbol{\lambda}_{21} < 0$. We are now in a position to introduce our findings shown in Result 6.

Result 6. For $r_i = 0$, presence of cost learning curve and CDG demand functions (39) satisfying properties (36) and (40)

(i) The ratio R_i of the advertising elasticity of demand of firm i $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ equals the ratio of advertising to sales revenue of firm i $(U_i / f_i\ P_i)$, multiplied by Q_i'.
(ii) Optimal price increases over time for each competitor i.
(iii) Optimal advertising is decreasing over time for each competitor i.
(iv) The ratio of the advertising elasticity of demand $(\xi_i = U_i f_{iUi}/f_i)$ to its price elasticity $(\Lambda_i = -P_i f_{ipi}/f_i)$ decreases over time for each competitor i, provided that $f_i/f_{iPi} + P_i > 0$.

Part (iii) of the proposition does not require the restriction $f_{iNi} > 0$ depicted in (40). Except for optimal price, Result 6 also generalizes Proposition 2(i) and Result 2 for a dynamic monopoly to a dynamic duopoly. In its entirety, however, the contents of Result 6 are inconsistent with those related to Result 5 and are novel to the literature. Interestingly, it is observed that Result 6 for CDG is consistent with Result 4 for FPS. However, the assumptions deriving related findings are different (assumptions (36) and (40) for Result 6 versus assumptions (21), (31a) and (33) for Result 4). Example 7 is introduced next to position the model used in Result 6 relative to its counterparts in the relevant literature.

Example 7. Thompson and Teng (1984) combine their earlier model of advertising competition (Teng & Thompson, 1983) and the pricing response function of Robinson and Lakhani (1975) model to produce the following marketing-mix new product oligopoly model:

$$d N_i/dt = f_i = [(\gamma_{i1} + \gamma_{i2}U_i)(M - N_1 - N_2) + (\gamma_{i3} + \gamma_{i4}U_i)(M - N_1 - N_2)N_i/M]\,\mathrm{Exp}\,(-\gamma_{i5}P_i),\ N_i(0) = N_{i0}.$$

The authors employ an advertising cost function given by $Q_i = \alpha_i U_i^2 + \beta_i U_i + \delta_i$, where α_i, β_i, δ_i and γ_{ik}, $k = 1, 2, \ldots, 5$ are positive parameters and arrive at their open loop optimal policies for a finite planning horizon using numerical methods. Upon putting $\gamma_{i1} = a_{i1}p$, $\gamma_{i2} = a_{i2}p$, $\gamma_{i3} = a_{i1}q$, $\gamma_{i4} = a_{i2}q$ where a_{i1}, a_{i2}, p and q are positive parameters such that $q > p$, the above demand functions would take the form $d\,N_i\,/\,dt = f_i = (p + q\ N_i\ /M)(M - N_1 - N_2)$ $\mathrm{Exp}\,(-\gamma_{i5}P_i)\,(a_{i1} + a_{i2}U_i)$.

The above demand functions can be cast into expression (39) satisfying properties (36) and the additional properties $f_{1N2} \le 0$ and $f_{2N1} \le 0$. For the remaining additional properties in (40); ($f_{1N1} \ge 0$ and $f_{2N2} \ge 0$) to get satisfied, it can be easily shown that N_1 and N_2 would satisfy the inequalities $2N_1 + N_2 \le M$ (q - p)/q, $N_1 + 2N_2 \le M$ (q - p)/q, $N_1 \ge 0$ and $N_2 \ge 0$. According to Result 6 and for both firms, the open loop pricing policies would be monotonically increasing over time and the open loop advertising policies would be monotonically decreasing over time.

Example 8. Krishnamoorthy et al. (2010) consider a differential game for a duopoly summarized as follows:

Find optimal trajectories P_1^* and U_1^* to maximize

$$\int_0^\infty e^{-r_1 t}\left[(P_1 - C_1)(d\,N_1/dt) - d_1\,U_1^2/2\right]dt.$$

Find optimal trajectories P_2^* and U_2^* to maximize

$$\int_0^\infty e^{-r_2 t}\left[(P_2\ -\ C_2)(d\ N_2/dt) - d_2\ U_2^2/2\ \right]dt.$$

$$\text{s.t.}\ d\,N_1/dt = \rho_1 P_1^{-\eta_1} U_1\sqrt{M - N_1 - N_2},\ N_1(0) = N_{10},$$
$$d\,N_2/dt = \rho_2 P_2^{-\eta_2} U_2\sqrt{M - N_1 - N_2},\ N_2(0) = N_{20}.$$

where M, C_1, C_2, d_1, d_2, η_1, η_2 are positive constants. The authors only develop Nash equilibrium feedback strategies for a duopoly using the HJB equations and demonstrate that optimal price is constant over time for both firms whereas optimal advertising is declining over time for both firms and proportional to $\sqrt{M - N_1 - N_2}$. When the pricing response functions $P_i^{-\eta_i}$ are replaced with their linear counterparts α_i - β_i P_i and α_i and β_i are positive parameters, $i = 1, 2;$ the characterization of the optimal feedback strategies remain unchanged.

To arrive at open loop Nash equilibrium, we apply (37) and find that for small discounting rates ($r_1 = r_2 = 0$), the sign of optimal price of Firm i has the ambiguous sign of $f_{iNi} f_i - \lambda_{ij} f_{iPi} f_{jNi}$ whereas optimal advertising would be decreasing over time. The optimal feedback strategies developed by the authors appear different from the open loop strategies characterized above.

7. Summary and conclusions

This section summarizes the main theoretical findings of the study, highlights their managerial implications, and proposes directions for future research. The dynamic models analytically explored in this article represent a unique attempt in the literature aiming at characterizing over time optimal pricing and advertising policies together with the optimal ratio of advertising elasticity of demand to its price elasticity for two broad classes of products and services.

7.1. Summary of results

This paper has focused on a series of monopolistic and duopolistic dynamic marketing-mix models for frequently purchased products and services (FPS) as well as consumer durable goods (CDG). Our approach was to derive results analytically, rather than by numerical methods, trying to maintain a certain degree of generality (flexibility). For competitive models, optimal marketing–mix strategies are identified as open loop and feedback Nash solutions. A summary of various demand specifications used in this paper, and the assumptions on discount rates, cost learning and length of the planning horizon, is provided in Table 2. The body of the table depicts the optimal price, advertising and the ratio of advertising elasticity of demand to its price elasticity paths and the major assumptions (properties) made to derive the findings. In the columns of Table 2 we have indicated the type of demand functions used: (i) general, (ii) multiplicative separable with firm-specific adoption effects and (iii) other structures, in particular firm's own marketing-mix variables and adoption effects of all rivals. Our models contain three main dynamic elements. We introduced demand side learning effects within the frame of dynamic FPS through the acquisition and retention processes of frequently purchased products and subscription services and dynamic CDG through innovation, imitation, and saturation phenomena. The production technologies used by the firms were assumed to exhibit cost learning. Discounting is also considered because of its managerial relevance as it measures how profits earned sooner are compared to those earned later. Optimal marketing-mix policies for FPS appear different from those related to CDG in both monopolistic markets (Proposition 1 versus Proposition 2) and duopolistic markets (Proposition 3 versus Proposition 4). However, the ratio of advertising elasticity of demand to its price elasticity appears to have been governed by similar set of rules (Proposition 1(i) and Proposition 2(i) for a monopoly, and Proposition 3(i) and Proposition 4(i) for a duopoly).

Results 1, 3, and 4 for FPS associated with low discounting argue in favor of a pricing strategy that is increasing over time and an advertising strategy that is decreasing over time. These dominant policies deserve an explanation. The agent (seller/ service provider) charges a low price/service fee and advertises heavily at the beginning to generate positive word of mouth, speed up the learning of the agent's offering and build a large base of purchasers/adopters who keep on purchasing the product/service upon their satisfaction with their initial experiences, causing the perceived value of the product/service to get enhanced over time. Also, because revenues of the agent in a given time period are not solely generated from new purchasers/adopters during that period but also from purchasers/adopters in previous periods, price/subscription fee is motivated to increase and advertising is invigorated to decrease in order to increase profits (Mesak & Darrat, 2002; Nagle, 1987).

Results 2 and 5 for CDG associated with low discounting argue in favor of a pricing strategy that mimics the rate of change of de-

Table 2
Summary of models, assumptions and results.

Demand function	Properties	Results	Market structure Solution concept	Type of good	Cost learning	r_i	Planning horizon
$f(N, P, U)$ General	(2)	Proposition 1 $R=(U/NP)\,Q'$ $\dot{P}, \dot{U}$: Prop. 1 (ii) $\dot{R}$: Prop. 1 (iii)	Monopoly Pontryagin's Maximum principle	FPS	Present	≥0	Finite/ Infinite
$f(N, P, U)$ General	(15)	Proposition 2 $R=(U/fP)\,Q'$ $\dot{P}, \dot{U}$: Prop. 2 (ii) $\dot{R}$: Prop. 2 (iii)	Monopoly Pontryagin's maximum principle	CDG	Present	≥0	Finite/ Infinite
$f_i(N_i, N_j\ (P_i, U_i), (P_j, U_j))$ General	(21)	Proposition 3 $R_i = \frac{(Q'_i - \lambda_{ij}\ f_{jUi}\ )U_i}{(N_i + \lambda_{ij} f_{jPi})P_i}$ $\dot{P}_i, \dot{U}_i$: Prop. 3 (ii) $\dot{R}_i$: Prop. 3 (iii)	Duopoly Open- loop Nash equilibrium	FPS	Present	≥0	Finite/ Infinite
$f_i(N_i, N_j\ (P_i, U_i), (P_j, U_j))$ General	(36)	Proposition 4 $R_i = \frac{(Q'_i - \lambda_{ij}\ f_{jUi}\ )U_i}{(N_i + \lambda_{ij} f_{jPi})P_i}$ $\dot{P}_i, \dot{U}_i$: Prop. 4 (ii) $\dot{R}_i$: Prop. 4 (iii)	Duopoly Open- loop Nash equilibrium	CDG	Present	≥0	Finite/Infinite
$f=g(N)\ h\ (P,U)$	(12)	Result 1 $R=(U/NP)Q'$ $\dot{P} > 0$, $\dot{U} < 0$ $\dot{R} < 0$	Monopoly Pontryagin's Maximum principle	FPS	Present	0	Finite/Infinite
$f=g(N)\ h\ (P,U)$	(18)	Result 2 $R=(U/fP)Q'$ $\dot{P} = \text{Sign}\ f_N$, $\dot{U} = 0$ $\dot{R} = \text{Sign} - f_N$	Monopoly Pontryagin's maximum principle	CDG	Present	0	Finite/Infinite
$f_i(N_i, (P_i, U_i), (P_j, U_j))$	(21), (31), (31a)	Result 3 $R_i=(U_i/N_iP_i)Q'_i$ $\dot{P}_i>0$, $\dot{U}_i < 0$ $\dot{R}_i < 0$	Duopoly Open-loop Nash equilibrium	FPS	Present	0	Finite/Infinite
$f_i(N_i, N_j\ (P_i, U_i))$	(21), (31a), (33)	Result 4 $R_i=(U_i/N_iP_i)Q'_i$ $\dot{P}_i>0$, $\dot{U}_i < 0$ $\dot{R}_i < 0$	Duopoly Open-loop Nash equilibrium	FPS	Present	0	Finite/Infinite
$f_i= g_i(N_i)\ h_{ii}(P_i)\ h_{jj}(P_j)$. $q_{ii}(U_{i)}q_{ij}(U_j)$	(36)	Result 5 $R_i=(U_i/f_iP_i)Q'_i$ $\dot{P}_i = \text{Sign}\ f_{iNi}$, $\dot{U}_i = 0$ $\dot{R}_i = \text{Sign} - f_{iNi}$	Duopoly Open-loop Nash equilibrium	CDG	Present	0	Finite/Infinite
$f_i= g_i(N_i,N_j)\ h_i(P_i)\ q_i(U_i)$	(36), (40)	Result 6 $R_i=(U_i/f_iP_i)Q'_i$ $\dot{P}_i > 0$, $\dot{U}_i < 0$ $\dot{R}_i < 0$	Duopoly Open-loop Nash equilibrium	CDG	Present	0	Finite/Infinite
$f=g(N)Cos(\theta P)U$	(B1a)	Result 7 $R=(U/NP)Q'$ $\dot{P}_i = 0$, $\dot{U} > 0$ $\dot{R}_i = 0$	Monopoly HJB equation	FPS	Not Present	0	Infinite
$f_i= \gamma_i\text{Cos}\ (\theta_iP_i)$. $\text{Cos}\ (\theta_jP_j)\ U_iU_jg(N)$	(B14a)	Result 8 $R_i=(U_i/N_iP_i)Q'_i$ $\dot{P}_i = 0$, $\dot{U}_i > 0$ $\dot{R}_i = 0$	Duopoly Feedback Nash equilibrium	FPS	Not Present	0	Infinite

mand with respect to penetration over time (increasing first and declining later) and an advertising strategy that is constant over time. These dominant policies also deserve an explanation. Kalish (1983) mentions that for a period of positive effects of sales on demand, price is initially low to stimulate early adopters, which in turns will stimulate demand. Price will monotonically increase to the point where the "word-of-mouth" effect diminishes. On the other hand, if there is a negative effect of sales now on subsequent demand, price is initially relatively high, skimming some profits from those who are willing to pay for early adoption, decreasing monotonically over time. When the demand functions are taken to be multiplicatively separable in advertising and the number of cumulative adopters, advertising elasticities are unaffected by changes in sales. So that advertising should be kept constant over time (Dockner & Jørgensen, 1992; Teng & Thompson, 1985). To better comprehend the contents of four propositions and eight results eight illustrative examples, mainly extracted from the relevant literature, are introduced.

For separable multiplicative FPS demand functions and in the absence of discounting, the feedback marketing-mix strategies developed in this paper for both monopolistic and duopolistic markets are such that optimal price would be constant and advertising would be increasing over time. On the other hand, for separable multiplicative CDG demand functions, Sethi et al. (2008) and Krishnamoorthy et al. (2010) find for both monopolistic and duopolistic markets that the feedback marketing-mix strategies are such that optimal price would be constant, but optimal advertising would be decreasing over time (see Examples 3 and 8 for details). In Table 2, while the ratio of advertising elasticity to price elasticity appears to have been governed by similar set of rules for both FPS and CDG as in the extant literature (e.g., Mesak & Clark, 1998; Dockner and Feichtinger, 1986) the direction of change of such ratio over time looks, however, different from each other which is a new result to the literature (we are indebted to an anonymous reviewer for motivating us to pursue the above research direction).

7.2. Managerial implications

The problem studied in this research is briefly stated as follows: A firm in a monopoly or a duopoly seeks an optimal price and advertising over time to maximize its discounted profits, given that the competitor when being present acts rationally. Unit costs decreases with cumulative output and the current sales of each firm depend on the prices and advertising expenditures of all goods as well as firms' cumulative sales (subscriptions). Asserting that model assumptions are met three main recommendations standout particularly for low interest rates.

First, marketing-mix policies for new frequently purchased products and services (FPS) are different from their counterparts for new consumer durable goods (CDG). Second, for FPS, the research findings depicted in Table 2 recommend increasing price, decreasing advertising and a non-increasing ratio of advertising to sales revenue over time. Only, *the direction of change* of marketing-mix variables and the elasticities ratio is recommended to carry over from a monopoly to a duopoly for short planning horizons where open-loop Nash equilibrium is most appealing, but are not necessarily being similar in terms of *level*. For long planning horizons where feedback strategies are most useful, a constant price and increasing advertising over time are recommended to carry over from monopoly to a duopoly (Results 7 and 8).

Third, for CDG and in a monopoly, the research findings depicted in Table 2 argue in favor of a price increase when adoption effect is positive due to positive word of mouth and a price decrease when adoption effect is negative attributed to saturation (i.e., remaining untapped market is depleting). Advertising expenditure is recommended to be constant over time, whereas the ratio of advertising to sales revenue would change in an opposite direction of the change in price. For duopolistic markets and for short planning horizons, monopoly marketing strategies may (or may not) be recommended to carry over from a monopoly to a duopoly depending upon how management envisions the likely demand functions would be (Result 5 versus Result 6). For long planning horizons, a constant price and a decreasing advertising over time are recommended to carry over from a monopoly to a duopoly (Examples 3 and 8).

7.3. Suggestions for future research

The modeling effort developed here though being extensive is susceptible to further improvement through future research. First, other controllable variables such as product/ service quality (Rust, Zahorick & Keiningham, 1995; Van Mieghem, 2000), distribution (Bronnenberg, Mahajan & Vanhonacker, 2000; Jones & Ritz, 1991); and inventory (Mesak, Bari & Blackstock, 2016) could be endogenously determined and thus in turn enriching the modelling effort. Second, throughout the paper we have considered monopolistic markets, or duopolistic markets supposing that all firms are in the market from the very beginning of the game. In reality, the market is generally monopolistic at first and becomes competitive when new rivals get in. Hence the well-known problem of entry of rivals emerges. Future research can enlarge the scope of the study by considering such problem in the modelling framework (Eliashberg & Jeuland, 1986; Gupta and Di Bendetto, 2007). Third, in this article competition was limited to duopolistic markets. Future research would extend the study to deal with oligopolistic competition. Studying symmetric competition (Wernerfelt, 1986) may be a reasonable first approximation.

Studies in rivalry advertising in the context of social networks including advertising competition in social networks (Masucci and Silva (2017), competitive targeted advertising over networks (Bimpikis, Ozdaglar & Yildiz, 2016), and control of preferences in social networks (Chasparis and Shamma (2010). The incorporation of additional control variables in the context of social networks represents an additional direction for future research. Although differential games with feedback strategies are notoriously hard to analyze to arrive at optimal marketing-mix strategies, interested readers could find the progress being made in this article as well as in other articles helpful in advancing the state of the art. Addressing the above research issues and perhaps several others, should be beneficial to both academicians and practitioners in both the manufacturing and the service sectors.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2019.07.040.

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