Chapter 4

Stationarity and Unit Roots Tests

4.1   Introduction

Many financial time series, like exchange rate levels of stock prices appear to be non-stationary. New statistical issues arises when analyzing non-stationary data. Unit root tests are used to detect the presence and form of non-stationarity.

This chapter reviews main concepts of non-stationarity of time series and provides a description of some tests for time series stationarity. More information about such tests can be found in Hamilton (1994), Fuller (1996), Enders (2004), Harris (1995), Verbeek (2008).

There are two principal methods of detecting nonstationarity:

- Visual inspection of the time series graph and its correlogram;

- Formal statistical tests of unit roots.

We will start with formal testing procedures first.

A nonstationaty time series is called *integrated* if it can be transformed by first differencing once or a very few times into a stationary process. The *order of integration* is the minimum number of of times the series needs to be first differenced to yield a stationary series. An integrated of order 1 time series is denoted by $I(1)$. A stationary time series is said to be integrated of order zero, $I(0)$.

4.2   Unit Roots tests

Let us consider a time series Y_t in the form

$$Y_t = \alpha + \beta Y_{t-1} + u_t$$
$$u_t = \rho u_{t-1} + \varepsilon_t \qquad\qquad (4.2.1)$$

Unit root tests are based on testing the null hypothesis that $H_0\colon \rho = 1$ against the alternative $H_1\colon \rho < 1$. They are called unit root tests because under the null hypothesis the characteristic polynomial has a root equal to unity. On the other hand, stationarity tests take the null hypothesis that Y_t is trend stationary.

4.2.1   Dickey-Fuller test

One commonly used test for unit roots is the Dickey-Fuller test. In its simplest form it considers a $AR(1)$ process

$$Y_t = \rho Y_{t-1} + u_t$$

where u_t is an IID sequence of random variables. We want to test

$$H_0\colon \rho = 1 \text{ vs. } H_1\colon \rho < 1.$$

Under the null hypothesis Y_t is non-stationary (random walk without drift). Under the alternative hypothesis, Y_t is a stationary $AR(1)$ process.

Due to non-stationarity of Y_t under the null, the standard t-statistic does not follow t distribution, not even asymptotically. To test the null hypothesis, it is possible to use

$$DF = \frac{\hat{\rho} - 1}{s.e\,(\hat{\rho})}.$$

Critical values, however, have to be taken from the appropriate distribution, which is under the null hypothesis of non-stationarity is nonstandard. The asymptotic critical values of DF based on computer simulations are given in Fuller (1996).

More general form of the Dickey-Fuller test employs other variants of the time series process. Consider the following three models for the data generating process of Y_t:

$$Y_t = \rho Y_{t-1} + u_t \tag{4.2.2}$$

$$Y_t = \rho Y_{t-1} + \alpha + u_t \tag{4.2.3}$$

$$Y_t = \rho Y_{t-1} + \alpha + \beta t + u_t \tag{4.2.4}$$

with u_t being iid process.

Dickey and Fuller (1979) derive a limiting distribution for the least squares t-statistic for the null hypothesis that $\rho = 1$ and F-statistic (Wald statistic) for the null hypotheses of validity of combinations of linear restrictions $\rho = 0$, $\alpha = 0$ and $\beta = 0$ where the estimated models are from (4.2.2) to (4.2.4) but in each case that (4.2.2) is the true data generating process.

4.2.2 Augmented Dickey-Fuller test

Dickey and Fuller (1981) show that the limiting distributions and critical values that they obtain under the assumption of iid u_t process are also valid when u_t is autoregressive, when augmented Dickey-Fuller (ADF) regression is run. Assume the data are generated according to (4.2.2) with $\rho = 1$ and that

$$u_t = \theta_1 u_{t-1} + \theta_2 u_{t-2} + ... + \theta_p u_{t-p} + \varepsilon_t \qquad (4.2.5)$$

where ε_t are iid. Consider the regression

$$\Delta Y_t = \phi Y_{t-1} + \alpha + \beta t + u_t$$

and test $H_0\colon \phi = 0$ versus $H_1\colon \phi < 0$. Given the equation for u_t in (4.2.5) we can write

$$\Delta Y_t = \phi Y_{t-1} + \alpha + \beta t + \theta_1 u_{t-1} + \theta_2 u_{t-2} + ... + \theta_p u_{t-p} + \varepsilon_t.$$

Since under $\rho = 1$ we have $u_t = Y_t - Y_{t-1}$, this equation can be rewritten as

$$\Delta Y_t = \phi Y_{t-1} + \alpha + \beta t + \theta_1 \Delta Y_{t-1} + \theta_2 \Delta Y_{t-2} + ... + \theta_p \Delta_{t-p} + \varepsilon_t. \qquad (4.2.6)$$

Said and Dickey (1984) provide a generalization of this result for $ARMA(p, q)$ error terms.

Procedure Before using the ADF test we have to decide how many lags of ΔY to include in the regression. This can be done by sequentially adding lags and testing for serial correlation using Lagrange multiplier tests to archive a white noise residuals.

Use F-test to test the null $(\beta, \rho) = (0, 1)$ against the alternative $(\beta, \rho) \neq (0, 1)$. If the null is rejected we know that

$$\text{either } \begin{bmatrix} \beta \neq 0 \\ \rho = 1 \end{bmatrix} \text{ or } \begin{bmatrix} \beta = 0 \\ \rho \neq 1 \end{bmatrix} \text{ or } \begin{bmatrix} \beta \neq 0 \\ \rho \neq 1 \end{bmatrix}$$

and the next step is to test $\rho = 1$ using the t-statistic obtained from the estimating the augmented version of (4.2.4), with the critical values taken from the standard normal tables. Critical values from the standard normal are appropriate when β is non-zero, so that if the null hypothesis is not rejected we can rule out the second and third cases (if β is zero the critical values are non-standard, but will be smaller than the standard normal ones). Thus, if $\rho = 1$ is accepted we conclude that $\beta \neq 0$ and $\rho = 1$, so that series has a unit root and a linear trend.

If we reject the null then the first alternative can be dismissed. This leaves the following two alternatives

$$\text{either } \begin{bmatrix} \beta = 0 \\ \rho \neq 1 \end{bmatrix} \text{ or } \begin{bmatrix} \beta \neq 0 \\ \rho \neq 1 \end{bmatrix}$$

In either case ρ is not 1, there is no unit root and conventional test procedures can be used. Thus we may carry out a t test for the null that $\beta = 0$.

If we cannot reject $(\beta, \rho) = (0, 1)$ we know that the series has a unit root with no trend but with possible drift. To support the conclusion that $\rho = 1$ we may test this, given β is assumed to be zero.

If we wish to establish whether the series has non-zero drift, further tests will be required. Note that we know $(\beta, \rho) = (0, 1)$, and so we might carry out the F test. This tests

$$H_0 \colon (\alpha, \beta, \rho) = (0, 0, 1) \text{ vs. } (\alpha, \beta, \rho) \neq (0, 0, 1).$$

If we cannot reject the null hypothesis, the series is random walk without drift. If we reject it, the series is a random walk with drift.

We may wish to support these findings on the basis of estimating (4.2.3) by setting β at zero as suggested by the various previous tests. If β is actually zero then tests on α and ρ should have greater power once this this restriction is imposed.

4.2.3   Phillips and Perron tests

The statistics proposed by Phillips and Perron (1988) (Z statistics) arise from their considerations of the limiting distributions of the various Dickey-Fuller statistics when the assumption that u_t is an iid process is relaxed.

The test regression in the Phillips-Perron test is

$$\Delta Y_t = \phi Y_{t-1} + \alpha + \beta t + u_t$$

where u_t is a stationary process (which also may be heteroscedastic). The PP tests correct for any serial correlation and heteroscedasticity in the errors ut of the test regression by directly modifying the test statistics. These modified statistics, denoted Z_t and Z_ϕ, are given by

$$Z_t = \left(\frac{\hat{\sigma}^2}{\hat{\lambda}^2}\right)^{\frac{1}{2}} t_{\phi=0} - \frac{1}{2}\left(\frac{\hat{\lambda}^2 - \hat{\sigma}^2}{\hat{\lambda}^2}\right)\left(\frac{T s.e\left(\hat{\rho}\right)}{\hat{\sigma}^2}\right)$$

$$Z_\phi = T\phi - \frac{1}{2}\left(\frac{T^2 s.e\left(\hat{\rho}\right)}{\hat{\sigma}^2}\right)\left(\hat{\lambda}^2 - \hat{\sigma}^2\right)$$

The terms $\hat{\sigma}^2$ and $\hat{\lambda}^2$ are consistent estimates of the variance parameters

$$\hat{\sigma}^2 = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} E\left[u_t^2\right]$$

$$\hat{\lambda}^2 = \lim_{T\to\infty} \sum_{t=1}^{T} E\left[T^{-1} S_T^2\right]$$

where $S_T = \sum_{t=1}^{T} u_t$.

In the Dickey-Fuller specification we can use the critical values given by Dickey and Fuller for the various statistics if u_t is an iid and we should use Phillips-Perron's counterparts if it is not iid.

An indication as to whether the Z statistic should be used in addition to (or instead of) the ADF tests might be obtained in the diagnostic statistics from the DF and ADF regressions. If normality, autocorrelation or heterogeneity statistics are significant, one might adopt the Phillips-Perron approach. Furthermore, power may be adversely affected by misspecifying the lag length in the augmented Dickey-Fuller regression, although it is unclear how far this problem is mitigated by choosing the number of lags using data-based criteria, and the Z-tests have the advantage that this choice does not have to be made. Against this, one should avoid the use of the Z test if the presence of negative moving average components is somehow suspected in the disturbances.

Under the null hypothesis that $\phi = 0$, the PP Z_t and Z_ϕ statistics have the same asymptotic distributions as the ADF t-statistic and normalized bias statistics. One advantage of the PP tests over the ADF tests is that the PP tests are robust to general forms of heteroskedasticity in the error term u_t. Another advantage is that the user does not have to specify a lag length for the test regression.

4.3   Stationarity tests

The ADF and PP unit root tests are for the null hypothesis that a time series Y_t is $I(1)$. Stationarity tests, on the other hand, are for the null that Y_t is $I(0)$. The most commonly used stationarity test, the KPSS test, is due to Kwiatkowski, Phillips, Schmidt and Shin (1992) (KPSS). They derive their test by starting with the model

$$Y_t = \alpha + \beta t + \mu_t + u_t$$

$$\mu_t = \mu_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$$

where u_t is $I(0)$ and may be heteroskedastic.

The null hypothesis that Y_t is $I(0)$ is formulated as $H_0 \colon \sigma_\varepsilon^2 = 0$, which implies that μ_t is a constant. Although not directly apparent, this null hypothesis also implies a unit moving average root in the ARMA representation of ΔY_t.

The KPSS test statistic is the Lagrange multiplier (LM) or score statistic for testing $\sigma_\varepsilon^2 = 0$ against the alternative that $\sigma_\varepsilon^2 > 0$ and is given by

$$KPSS = \left(\frac{1}{T^2} \sum_{t=1}^{T} \hat{S}_t^2 \right) / \hat{\lambda}^2$$

where $\hat{S}_t = \sum_{j=1}^{t} \hat{u}_j$, $\hat{u}_t$ is the residual of a regression Y_t on t and $\hat{\lambda}_t^2$.

Critical values from the asymptotic distributions must be obtained by simulation methods. The stationary test is a one-sided right-tailed test so that one rejects the null of stationarity at the α level if the KPSS test statistic is greater than the $100(1-\alpha)$ quantile from the appropriate asymptotic distribution.

4.4   Example: Purchasing Power Parity

It is very easy to perform unit root and stationarity tests in EViews. As an example, consider a Purchasing Power Parity condition between two countries: USA and UK. In efficient frictionless markets with internationally tradeable goods, the law of one price should hold. That is,

$$s_t = p_t - p_t^*,$$

where s_t is a natural logarithm of the spot exchange rate (price of a foreign currency in units of a domestic one), p_t is a logarithm of the aggregate price index in the domestic country and p_t^* is a log price in the foreign country. This condition is referred to as absolute purchasing power parity condition.

This condition is usually verified by testing for non-stationarity of the real exchange rate $q_t = s_t + p_t^* - p_t$. Before we perform this let us look at properties of the constituent series.

We consider monthly data for USA and UK over the period from January 1989 to November 2008.

Although plots of both consumer price indices and the exchange rate indicate non-stationarity, we perform formal tests for unit root and stationarity.

In the dataset PPP.wf1, there are levels of the exchange rate and consumer price indices are given, so we need to create log series to carry out tests. This is done as usually,

    series lcpi_uk=log(cpi_uk)

    series lcpi_uk=log(cpi_uk)

    series lgbp_usd=log(gbp_usd)

Let us start with the UK consumer price index. We can find the option **Unit Root Tests...** in the **View** section of the series object menu (double click on lcpi_uk icon). In the **Test Type** box there is a number of tests available in EViews. We start with Augmented-Dickey-Fuller test. As we are interested in testing for unit roots in levels of log consumer price index, we choose **Test for unit root in levels** in the next combo-box, and finally we select testing with both intercept and trend as it is the most general case.
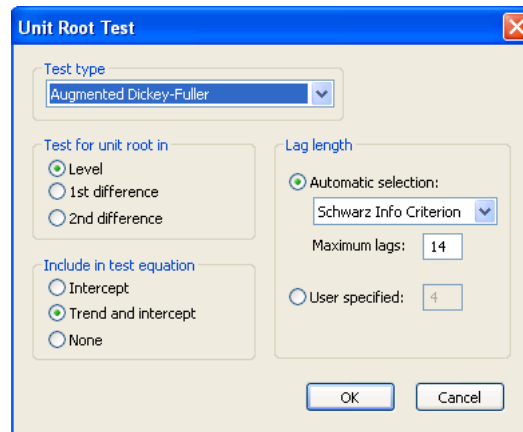
Figure 4.1: Augmented Dickey-Fuller test dialog window

EViews will also select the most appropriate number of lags of the residuals to be included in the regression using the selected criteria (it is possible to specify a number of lags manually is necessary by ticking **User specified** option).

Click **OK** and EViews produces the following output
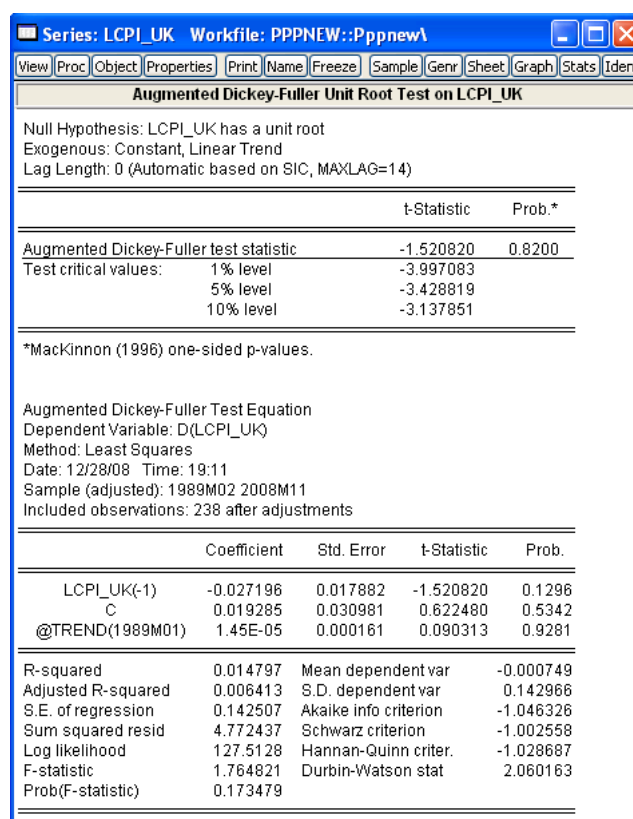
Figure 4.2: Output for the Augmented Dickey-Fuller test

The absolute value of the t-statistic does not exceed any of the critical values given below so we cannot reject the null hypothesis of the presence of unit root in the series.
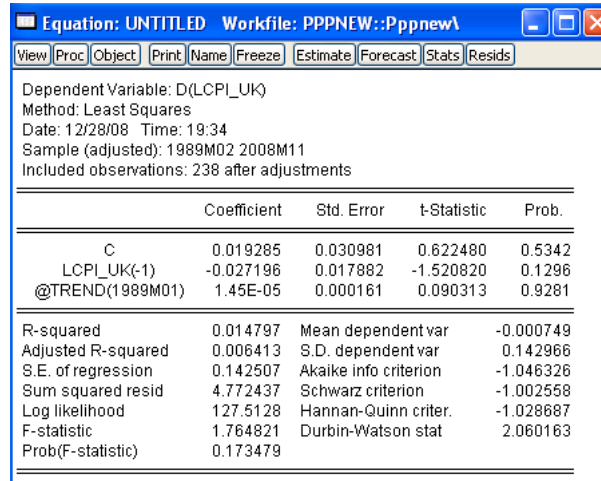
Unfortunately, EViews provides only the test of the null hypothesis $H_0\colon \phi = 0$. One can perform more general test by estimating Dickey-Fuller regression (4.2.6). In the command line type the following specification

    ls d(lcpi_uk) c lcpi_uk(-1) @trend(1989M01)

to run the ADF regression with intercept and trend component. As you noted, the function @trend allows to include the time trend component that increases by one for each date in the workfile. The optional date argument 1989M01 is provided to indicate the starting date for the trend. We did not include any MA components in

the regression since based on the previous results (see Figure **??**) zero lag is optimal according to the Schwartz selection criterium.

The regression output is identical with that produced by the Augmented Dickey-Fuller test

Figure 4.3: Output of the regression-based procedure of the Augmented Dickey-Fuller test

However, the approach enables us to perform the Wald test of linear restrictions and specify the null hypothesis $H_0\colon (\beta, \phi) = (0, 0)$ (or more general, $H_0\colon (\alpha, \beta, \phi) = (0, 0, 0)$).
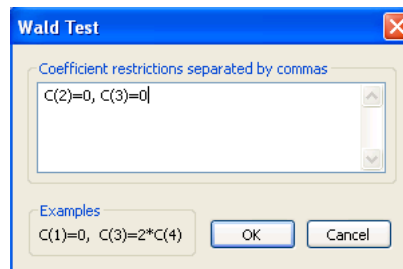
Figure 4.4: Wald test results for the Augmented Dickey-Fuller test specifications

The value of Wald test statistic in the case of the null $H_0\colon (\beta, \phi) = (0, 0)$ is 1.7648; this has to be compared with the critical values tabulated in MacKinnon (1996).

As the test statistic is smaller than all of the critical values, we cannot reject the null hypothesis, which confirms non-stationarity of the log consumer price index series.

This conclusion is also confirmed by other stationarity tests. For example, in Kwiatkowski-Phillips-Schmidt-Shin test (specification with the intercept and trend), the test statistic is 0.4257 which is higher than the critical value at 1% significance level (which is 0.216). Thus, we reject the null hypothesis of stationarity of the series.

If the Purchasing Power Parity condition holds one would expect the real exchange rate $q_t = s_t - p_t + p_t^*$ to be stationary and mean reverting. The presence of unit root in the deviations series would indicate the existence of permanent shocks which do not disappear in a long run.

We create a series of deviations

d=lgbp_usd-lcpi_uk-lcpi_us

Augmented Dickey-Fuller does not reject the null hypothesis of the presence of unit root in the deviations series. Also, the value of Wald test statistic is 1.8844 indicates which confirms nonstationarity of the deviations from the Purchasing Power Parity condition.